ISSUE No.17

PRACTICAL MATHEMATICS

A QUARTERLY

WITH APPLICATIONS IN THE FIELD OF BUSINESS

MATHEMATICS OF FINANCE

Installment Finance

Contracts • Amortization
Perpetual Payments
Capitalized Cost

Money and Banking

Chain Rule • Cable Rates
Conversion of Foreign Monies
Forward Exchange
Discount of Bonds
Stock Purchase Warrants
Arbitrage

—ALSO—

Financial Tables and Formulas
Self-Tests and Mathematics Problems

PARK J. EWART, Ph.D.
University of Southern California



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EDITOR: REGINALD STEVENS KIMBALL ED.D.

Issue No.17 PRACTICAL MATHEMATICS Volume No.3

ISSUE 17 **Practical Mathematics** VOLUME 3

REGINALD STEVENS KIMBALL, *Editor*

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PRACTICAL MATHEMATICS is published quarterly by the National Education Alliance, Inc., Office of publication at Washington and South Avenues, Dunellen, N. J. Executive and editorial offices, 37 W. 47th St., New York 19, N. Y. John J. Crawley, President; A. R. Mahony, Vice President and Business Manager; Frank P. Crawley, Treasurer. Issue No. 17, May 20, 1944. Entered as second-class matter April 10, 1943, at the Post Office at Dunellen, N. J., under the Act of March 3, 1879. Printed in the U.S.A. Price in the U.S.A., 50c a copy; annual subscription at the rate of 50c a copy. *Contents copyright 1944 by National Educational Alliance, Inc.*

CHATS WITH THE EDITOR

THIRD in the series on mathematics applied to business, this issue is devoted to the mathematics of installment finance and of money and banking. These subjects follow naturally on the consideration which we have already given in Issue Fifteen to the matter of life insurance and in Issue Sixteen to the subject of accounting.

Most laymen are woefully ignorant of the processes by which finance companies arrive at a determination of the interest rates which they charge. When they find it necessary to arrange for partial payments, extending over a period of time, in connection with the purchase of an automobile or of a rather expensive item of household equipment, they realize that "something extra" has to be paid over and above the cash price. "A dollar down and a dollar a week" sounds harmless and is easily within the reach of the average pocketbook. Accepting the "service charge" which is added to the asking price, they seldom stop to figure out the interest rate which they are thus asked to pay.

The trouble with such schemes is that the purchaser usually fails to compare the ultimate cost under such a plan with the amount which the article would have cost if he could simply have withdrawn his money from the bank to make complete payment at the time of purchase. Given two propositions, each involving a moderate down payment and settlement for the balance on easy terms, he is unable to determine which would be less costly for him in the long run. Dr. Ewart throws light on this subject, showing the reader how to compare the two propositions.

In all of his computations, Dr. Ewart has followed a definite pattern. The reader who keeps this pattern in mind will have little difficulty in training himself to think along the same lines when he is confronted by a new problem. The tendency for students working in this field is to look for the formula first. Such a plan of attack often leads to confusion and uncertainty.

Much of the confusion will be avoided if you follow the scheme of beginning by visualizing the problem. Remember that words used in financial problems are abstract symbols standing for real objects and real activities. Although we all know this, we still have a tendency to consider words as something in and of themselves. Using words without reference to the objects and activities to which they refer will almost invariably lead to difficulty. We should make a conscious effort to refer the problem back to events and experiences in real life.

In this connection, you will find that you improve your ability to visualize if you will read a considerable number of word problems, merely attempting to picture to yourself the specific activities indicated by the problem. When a problem is based on a credit instrument, you may find it helpful to record the main characteristics on a blank sheet of paper representing the credit instrument.

While in the early stages of your study, you should concentrate on visualizing the problem without scattering your attention by seeking a method by which it may be solved. One little boy was asked how he knew whether to multiply or to divide. "First I try *gozinter* and if that doesn't work I know I have to do *times*"

was his reply. Such trial-and-error methods rarely lead to a real understanding of the principles involved.

After you have read the exact wording and translated the known and the unknown situations into algebraic symbols, place these symbols on a line chart which shows graphically that time between events is involved. The various time-line charts strewn through Dr. Ewart's article are of this nature. In each instance, the symbols written on and below the line are of assistance in leading to an understanding of the problem.

Now consider again the algebraic symbols which you have indicated. Select a formula which contains these symbols. Substitute into this formula the values previously determined, looking up extended values in the tables which are related to the formula. (In several instances, Dr. Ewart shows you that you have a choice of tables for this purpose.)

You are now ready to solve the resulting equation for the required unknown or unknowns. Having found these, check the reasonableness of your answer by inserting them in the original formula.

In most problems in this issue, you will find that your work is greatly facilitated if a calculating machine is available. Many of the computations require extended multiplications or divisions, which are tedious and time-consuming unless you resort to mechanical means of solution. Since every operation with numbers affords another chance for error, you may avoid these errors and save your time for additional practice if you run your figures through a calculating machine.

If you do not have a machine available, however, you should give attention to the alternative solutions in which logarithms are employed. If you are already familiar with the use of logarithms (first presented in Issue Number Two on pages 88 to 100), we shall probably not have to "sell" you at this point on their desirability. If you have been postponing the day when you intend to perfect yourself in their use, let us suggest that that day is now at hand. Obviously, the slide rule (pages 101 to 114) may also be employed for some of these calculations, though the degree of accuracy may not be quite so great as we should desire for the present purpose.

In Dr. Trefftzs's article, you are introduced to the mathematics of investment analysis. Methods of making mathematical appraisals of the intrinsic value of securities have been developed almost entirely since 1929.

Dr. Trefftzs begins with definitions of terms frequently encountered in the financial pages of the newspapers. After studying the formulas relating to these terms, you will find yourself in a much better position to understand the "back pages" of the metropolitan press, the pages which remain occult mysteries to most readers. This section of his article is thus of considerable value apart from its mathematical implications.

That portion of Dr. Trefftzs's article which deals with the conversion of foreign currencies throws further light on a subject which frequently presents difficulties to an American reader. Because of conditions incident to the present war, the whole future of foreign exchange is somewhat indefinite at this moment. The illustrations have naturally had to be thrown back to pre-war days. Whatever the new units of money may be, the general principles of conversion will be the same, however, so that mastery of the material here presented will place you in a position to undertake conversion problems in any unit which may be developed.

If you have been visualizing the banker's life as an easy one, you may come to a somewhat different conclusion after you have performed

some of the calculations which are a part of the banker's daily stock in trade. Perhaps you have often wondered just how a bank makes money on its deals. The illustrative examples concerning arbitraging show a type of transaction which is perfectly safe, no risk of monies being involved, but conservatively profitable.

It is probable that few readers of PRACTICAL MATHEMATICS will have occasion to engage in arbitraging to any considerable extent. At some time or other, however, most of them will probably "dabble in stocks", even if they are prudent enough not to attempt to "play the market". While some people seem to have considerable luck in playing hunches, and while it must be admitted that human factors which are not mathematically analyzable often go into the making or losing of huge fortunes, it should be realized that, in the long run, the conservative investor is in a more favorable position if he can calculate quickly and accurately the amount which he stands to gain or lose by a given transaction.

Dr. Trefftzs shows how to calculate the present worth of a stock or bond, based on the amount of interest it earns and on the date of its maturity. From the simple formulas which he illustrates, one may easily determine for himself the value of any stock which he is contemplating buying.

Both of the articles in the present issue lay considerable stress on the use of tables. Following the practice which we have used in all of our issues, PRACTICAL MATHEMATICS presents suitable tables in a sufficient degree of completeness to permit of the calculations necessary to the solution of the problems presented as exercises for the reader. Since complete tables would run to many pages, it should be obvious that limitations of space in a magazine like this make it impossible for us to present all of the tables which are found necessary in the fields covered. If you are actually working in any of these fields, you will want to acquire complete tables.

Much of the mathematics which is employed in the present issue is of a simple nature. The formulas are new, but are manipulated in accordance with principles earlier developed. There is always a tendency when we are engaged in learning new formulas or new processes to take for granted our ability to continue to remember and use what we have previously learned. The human brain is a tricky machine, however, and most of us find that we sooner or later forget much of what we have previously learned unless we take pains to refresh our memories from time to time. Psychologists tell us, fortunately, that we may quickly re-learn what we have once known and that we may decrease the curve of forgetting if we engage in periodic practice, at first at frequent intervals but gradually with a greater spread between periods.

Let me suggest to you that a careful review of the earlier issues of PRACTICAL MATHEMATICS is desirable at this time. If you have not looked back over these earlier issues in some little time, perhaps now is the opportune moment for you to begin a painstaking review. At several points in the present issue, our authors suggest references to material presented earlier in the course. If you were using this book as a text in the classroom, your instructor would undoubtedly see to it that you looked up these references and that you became letter perfect on the sections to which you had been referred. Since in this course you are your own taskmaster in this respect, it is up to you, the teacher, to see that you, the student, do not encounter difficulties at this stage because of your failure to relate this material properly to what has gone before.

R.S.K.

ABOUT OUR AUTHORS

PARK J. EWART, a Missourian by birth, has spent most of his life in California. After attending high school at Pomona, California, he majored in economics, chemistry, and mathematics at Pomona College, Claremont, California, receiving the degree of Bachelor of Arts *cum laude* in 1923. Spending the next two years at the Graduate School of Business Administration at Harvard University, he obtained the degree of Master of Business Administration in 1925.

As Associate in the Extension Division and the Bureau of Business Research at Iowa State University, Mr. Ewart made trade and industrial surveys of various Iowa cities and towns, the field work in each community being conducted in coöperation with the local Chamber of Commerce. Two years of service as Chief Accountant of the Western Operating Properties of the Associated Telephone Utilities Company, in Los Angeles and Long Beach, California, gave him further practical experience in statistical work.

In September, 1930, Mr. Ewart became Instructor in Banking and Finance at the University of Southern California, the institution with which he is still connected. Since September, 1936, his rank has been Assistant Professor of Finance. In 1940, he became Head of the Department of Finance and, in 1943, he was appointed Management Coördinator of the Engineering, Science, and Management War Training program.

After completing the residence requirement for the degree of Doctor of Philosophy in Economics at the University of California (Berkeley), he pursued graduate studies at the University of Southern California (Los Angeles), receiving the degree of Doctor of Philosophy in Education for his thesis on "Problems of Collegiate Business Education in the United States".

Dr. Ewart is a member of the Department of Business Education of the National Education Association, of the American Economic Association, of the American Association of University Teachers of Insurance, and of the American Academy of Political and Social Science. He is a member of four honorary scholastic fraternities: Phi Beta Kappa, Phi Delta Kappa, Beta Gamma Sigma, and Alpha Kappa Psi.

Dr. Ewart has contributed articles on insurance and credit to such publications as the *Southern California Business Review* and the *Insurance Journal of Southern California.*

KENNETH LEWIS TREFFTZS obtained the degree of Doctor of Philosophy from the University of Illinois in 1939, after two years as a Fellow in economics. He served as assistant economist to the Committee on Banking Research of the Illinois Bankers' Association during 1937-38 and as instructor in the Department of Economics at Carnegie Institute of Technology from 1939 to 1941. He has been an Assistant Professor of Banking and Finance at the University of Southern California since 1941.

Dr. Trefftzs is a member of the American Statistical Association, of the American Economic Association, and has been Assistant Secretary of the American Finance Association since 1941. He is also a member of Omicron Delta Gamma and of Beta Gamma Sigma.

Among Dr. Trefftzs's writings are many articles which have appeared in *Barron's*, the national financial weekly, and contributions to the *Journal of Political Economy* and the *Southern California Law Review.*

COURSE 3 *Practical Mathematics* PART 17

MATHEMATICS OF INSTALLMENT FINANCE

By Park J. Ewart, Ph.D.

ONE of the most startling developments in finance during the past two decades has been the rapid and extensive growth in volume and variety of installment credit. Each new type of installment contract gives rise to a new set of financial problems which mathematics may aid in solving. To those merchants and financiers who have successfully applied the principles of installment finance the rewards have been great. Many of those consumers who have not understood the installment contracts into which they have entered have paid excessive costs when less expensive plans were readily available. When we have to choose between two or more courses of action based upon the interpretation of the relative advantages of two or more actual or contemplated contracts or other financial arrangements, it is often necessary that each be framed on a comparable basis before we can make a rational decision. Variation between patterns of relation of sums of money in time leads to non-comparability between installment plans.

From one point of view, most mathematical problems in finance are concerned with the relations between sums of money in time. These problems, therefore, necessitate the recognition of values which are future to other values. Values which occur earlier chronologically than future values are referred to as present values. Perhaps as much confusion in interpreting word problems arises from the inability to differentiate between present and future values as from any other aspect of time placement of events. Many failures of economists and business men to give adequate solutions to problems in real life have been due to failure to grasp properly the relationship of events in time.

Space relations are easier to grasp than time relations because continual adjustment to objects in space is necessary to survival. It has been found advantageous, therefore, to picture the sequence and separation of events in time by the use of the line chart. Statisticians and historians have been utilizing this device quite successfully for a long time. The device is beginning to appear more frequently in books in mathematics of finance.

For example, it may appear that $50 paid in 4 installments at quarterly intervals is equal to $200, but a line chart like Fig. 1a shows the 4 pay-

ments of \$50 separated by 3-months' intervals. If one has to wait 3 months for the first \$50, 6 months for the next, and so on, the 4 payments of \$50 are worth less than \$200 now. The future, S, Fig. 1b, and the present, P, Fig. 1c, each show lump-sum payments of \$200 but made at entirely different times. Thus, while each may be thought of as \$200, they represent two separate and distinct orders of \$200.

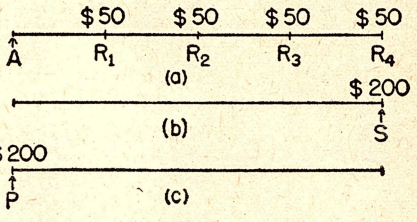


Fig. 1

In addition to showing the spread of events, these diagrams also may aid considerably in differentiating between present and future values. It is generally accepted practice to consider sums to the right as future to those to the left. A and P, then, in the design above, are present values and S is future. R_1 is present value to R_4; R_3 is future value to R_2; likewise, R_4 is future to R_3, etc.

INSTALLMENT FINANCE CONTRACTS

Installment financial contracts may be divided conveniently into annuities certain and contingent annuities, as was shown on pp. 900ff. Pages 903 to 917 then proceeded to develop systematically contingent annuities. We shall now consider a systematic treatment of annuities certain or mathematics of installment finance.

Mathmatical formulas may be classified into:

a Installment payments at end of period;
b Installment payments at beginning of period;
c Installment payments which are deferred or terminated one or more periods.

Each of these basic types of formula may be simplified if the rent period, or period between installments, is equal to the period of interest conversion. Therefore, each of the three basic types of installment-payment problems are classified in one of the two categories:

a Installment-payment period and interest-conversion period coincide;
b Interest-conversion period and installment-payment period not equal.

In turn, the problems in each of the cases may be classified into present-value problems and future-value problems. For example, all contracts involving periodic payments involve either the present value or the future value of a series of payments. From another point of view, a series of payments such as one might have coming from an estate will not be worth so much now as the sum of the individual payments. On the other hand, if these series of payments are allowed to accumulate into a fund and interest is accumulated, then the total accumulated is more than the sum of the separate payments.

Furthermore, both present-value and future-value installment problems involve additional variables, which we may conveniently classify under five headings, as follows:

a the rent or periodic payment,
b the number of payments,
c the rate of interest per period,
d the interest-conversion period,
e the rent-conversion period.

It is apparent that, in problems where d and e are equal, they may be ignored and there are, for practical purposes, only three of these variables. If all of these variables are known for a specific problem, then either the future value or the present value may be determined. If the future value or the present value is known, and any four of these variables are known also, then the fifth may be determined.

INSTALLMENT PAYMENTS	INSTALLMENT PERIOD AND INTEREST-CONVERSION PERIOD	VALUE
AT END OF PERIOD	COINCIDE	PRESENT
		FUTURE
	NOT EQUAL	PRESENT
		FUTURE
AT BEGINNING OF PERIOD	COINCIDE	PRESENT
		FUTURE
	NOT EQUAL	PRESENT
		FUTURE
DEFERRED OR TERMINATED ONE OR MORE PERIODS	COINCIDE	PRESENT
		FUTURE
	NOT EQUAL	PRESENT
		FUTURE

Fig. 2

Installment payments at end of period—ordinary annuity

The mathematics of compound interest was explained in detail on pages 1047 and 1048. An introduction to the relationship between compound interest and annuity certain formulas was presented on pages 897 to 900. The next two sections of this issue show in detail the transition from compound-interest problem-solving devices to those of periodic-payment problems.

CASE 1. INSTALLMENT-PAYMENT PERIOD AND INTEREST-CONVERSION PERIOD COINCIDE

Such a problem as this might arise: A father has decided to deposit $1,000 semi-annually during the next two and one-half years in a building-and-loan association for his son's education. If the association pays $3\frac{1}{2}\%$ compounded semi-annually, how much should be available to the boy at the end of 30 months?

DERIVATION OF FUTURE-VALUE FORMULAS AND TABLE

This problem may be solved readily as five separate compound-interest problems and the five future sums added to obtain the required answer.

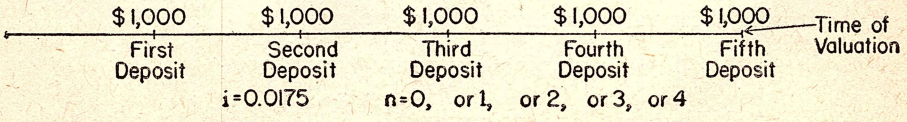


Fig. 3

Substituting these values in the compound-interest formula,

$$S=P(1+i)^n,$$

we obtain the following results:

Value of *last*	deposit at time of valuation	$\$1{,}000\ (1.0175)^0$
Value of *fourth*	deposit at time of valuation	$\$1{,}000\ (1.0175)^1$
Value of *third*	deposit at time of valuation	$\$1{,}000\ (1.0175)^2$
Value of *second*	deposit at time of valuation	$\$1{,}000\ (1.0175)^3$
Value of *first*	deposit at time of valuation	$\$1{,}000\ (1.0175)^4$.

Adding, we have

$$\$1{,}000(1.0175)^0+\$1{,}000(1.0175)^1+\$1{,}000(1.0175)^2+\$1{,}000(1.0175)^3+\$1{,}000(1.0175)^4.$$

After removal of the common factor, \$1,000, the future sum =

$$\$1{,}000[1+1.0175+(1.0175)^2+(1.0175)^3+(1.0175)^4].$$

When the number of deposits is small and an adding machine is available, this problem may be solved without a great amount of effort by adding the terms within the bracket, which may be found in Table LXXXIX. However, simpler methods have been developed:

For example, Table XCI records this summation of Table LXXXIX. Table XCI is merely the cumulative sum of the values in Table LXXXIX for 1 less than the number of periods plus \$1. For the solution of the illustrative problem, the value in Table XCI for $n=5$, $i=0.0175$, is \$5.17809. This is identical with the sum of the first four values in Table LXXXIX for $1\frac{3}{4}\%+\$1$; that is:

$$\$1+\$1.0175+\$1.03531+\$1.05342+\$1.07186=\$5.17809,$$

correct to the fifth place.

A similar summation will hold true for all values of n and all rates of interest in Tables LXXXIX and XCI. Table XCI will be of great value in solving installment finance problems pertaining to future values.

The symbol for values in Table XCI is $s_{\overline{n}|i}$, in which n stands for the number of periods and i stands for the rate of interest per period. This symbol is read "s, lower index n, at rate i". From another point of view, this is the symbol for the future value of \$1 per period for n periods at i rate of interest per period.

Likewise, it may be observed that

$$1+(1+i)+(1+i)^2+\ \ldots\ +(1+i)^{n-1}$$

is a geometric series in which

r(the rate of change) $=1+i$;
a(the first term) $=1$;
n = number of terms;
s = sum of a geometric series.

The formula for the sum of a geometric progression, as given on page 217, is

$$s=\frac{a(1-r^n)}{1-r}.$$

Substituting these values in this formula, we get

$$s_{\overline{n}|i}=\frac{1-(1+i)^n}{1-(1+i)}=\frac{1-(1+i)^n}{-i}=\frac{(1+i)^n-1}{i}.$$

This symbol, then, $s_{\overline{n}|i}$ and formula, $\frac{(1+i)^n-1}{i}$, both stand for the

values in Table XCI and also for the future value of \$1 per period for n periods at $i\%$.

Formulas for the future value of installments of the type discussed in this section, then, are as follows if

S_n = the future value

and R = the periodic payment:

$$S_n = R\, s_{\overline{n}|i} = R \cdot \frac{(1+i)^n - 1}{i}.$$

The uses of Table XCI symbol formula and the independent-variable formula will be demonstrated in detail later (pp. 1095 ff.). In the following section, we shall trace the derivation of Table XCII and the formulas for the present value of an annuity of \$1 per period for n periods at $i\%$ rate of interest from compound discount.

DERIVATION OF PRESENT-VALUE FORMULAS AND TABLE

At some time, one might be confronted with this proposition:

What is the value now of the right to receive \$1,000 semi-annually for the next five periods if interest is worth $3\frac{1}{2}\%$ compounded semi-annually?

This problem may be solved readily as five separate compound-discount problems and the five present values added to obtain the required answer.

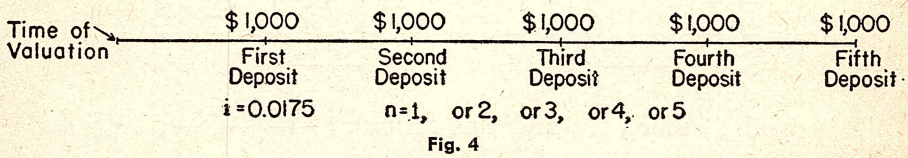


Fig. 4

Substituting these values in the compound-discount formula, $P = Sv_i^n$, we obtain the following results:

Value of *first* deposit at time of valuation = \$1,000 $v^1_{0.0175}$
Value of *second* deposit at time of valuation = \$1,000 $v^2_{0.0175}$
Value of *third* deposit at time of valuation = \$1,000 $v^3_{0.0175}$
Value of *fourth* deposit at time of valuation = \$1,000 $v^4_{0.0175}$
Value of *fifth* deposit at time of valuation = \$1,000 $v^5_{0.0175}$.

Adding, we have

$$\$1{,}000\, v^1_{0.0175} + \$1{,}000\, v^2_{0.0175} + \$1{,}000\, v^3_{0.0175} + \$1{,}000\, v^4_{0.0175} + \$1{,}000\, v^5_{0.0175}.$$

After removal of the common factor, the present value is

$$\$1{,}000(v^1_{0.0175} + v^2_{0.0175} + v^3_{0.0175} + v^4_{0.0175} + v^5_{0.0175}).$$

When the number of periodic deposits is small and an adding machine is available, this problem may be solved without a great amount of

effort by adding the terms within the bracket, which may be found in Table XC. However, simpler methods have been developed:

For example, Table XCII records this summation of values from Table XC. Table XCII is merely the cumulative sum of the values in Table XC for a given number of periods at a given rate of interest per period.

For the solution of the illustrative problem, the value in Table XCII for $n=5$, $i=0.0175$, is \$4.74786. This is identical with the sum of the first five values in Table XC for $1\frac{3}{4}\%$.

$$\$0.98280+\$0.96590+\$0.94929+\$0.93296+\$0.91691=\$4.74786.$$

Table XCII will be of great value in solving annuity-certain problems pertaining to present values.

The symbol for values in Table XCII is $a_{\overline{n}|i}$, in which n stands for the number of periods, and i stands for the rate of interest per period. This symbol is read "a, lower index n, at rate i". From another point of view, this is the symbol for the present value of \$1 per year for n periods at i rate of interest per period.

Likewise it may be observed that

$$\frac{1}{(1+i)^1}+\frac{1}{(1+i)^2}+\frac{1}{(1+i)^3}+\cdots+\frac{1}{(1+i)^{n-1}}+\frac{1}{(1+i)^n}$$

is a geometric series in which

$$r=\frac{1}{1+i}; \qquad a=\frac{1}{1+i}; \qquad n=\text{number of terms.}$$

Substituting these values in the formula for the sum of a geometric progression and using the symbol, $a_{\overline{n}|i}$, for the sum, we get

$$a_{\overline{n}|i}=\frac{\frac{1}{1+i}\left[1-\left(\frac{1}{1+i}\right)^n\right]}{1-\frac{1}{1+i}}=\frac{\frac{1}{1+i}\left[1-\frac{1}{(1+i)^n}\right]}{\frac{(1+i)-1}{1+i}}=\frac{1-(1+i)^{-n}}{i}.$$

This symbol, then, $a_{\overline{n}|i}$, and this formula, $\frac{1-(1-i)^{-n}}{i}$, both stand for the values in Table XCII and also for the present value of \$1 per period for n periods at $i\%$.

Formulas for the present value of annuities of the type to be discussed here, then, are as follows if

$A=$ the present value of series of rents of any size

and $R=$ the regular periodic payment:

$$A=Ra_{\overline{n}|i}=R\cdot\frac{1-(1+i)^{-n}}{i}=R\cdot\frac{1-\frac{1}{(1+i)^n}}{i}.$$

HOW TO USE FORMULAS AND TABLES

The appropriate uses for Tables XCI and XCII symbols and independent-variable formulas for both future and present values are illustrated in the following sections of Case 1.

Furthermore, it should be observed that in the future-value formulas

four variables were noted: S_n, R, n, i. Likewise, in the present-value formulas four variables were shown: A, R, n, i. In either of these two groups when three variables are known, the fourth may be found. We shall now proceed with detailed illustrations of how these formulas may be utilized in solving problems in which each of these variables is an unknown.

Amount unknown: future value—Mr. Smith has determined to deposit $500 at the end of each year for the next 10 years in a trust company. If the trust company pays 4% interest compounded annually, how much should Mr. Smith have credited to his account after the tenth deposit if no interest has been collected?

First, we translate the word problem into standard algebraic symbols:

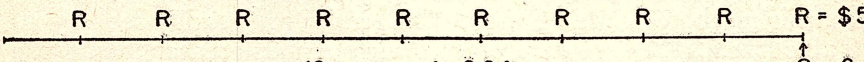


Fig. 5

The method that should be used in solving a given problem is the one which enables the answer to be obtained most easily and most accurately. If in a business situation one is confronted with a volume of problems requiring use of unusual interest rates[1] or periods, he should prepare a table or tables with these rates and periods.

Solution directly by Table XCI

If Table XCI is available and both the interest rate and number of periods in the specific problem may be found in the table, this method should be used:

a Select a formula which contains the table symbols:

$$S_n = R\, s_{\overline{n}|i}.$$

b Substitute the known variables shown in Fig. 5 into this formula.

$$S_n = \$500\, s_{\overline{10}|0.04}.$$

c From Table XCI:

$$S_n = (\$500)\ (12.00611).$$

d In this step, a calculating machine should be used if available.

$$S_n = \$6{,}003.06,$$

the amount Mr. Smith should have to his credit just after the tenth deposit.

Solution directly by Table LXXXIX

When both the interest rate and the number of periods in the specific problem may be found in Table LXXXIX, it may be used in lieu of Table XCI.

a Select a formula containing the independent-variable symbols:

$$S_n = R \cdot \frac{(1+i)^n - 1}{i}.$$

[1]Rates of interest and number of periods in the tables on pages 1169 to 1182 are rates and periods frequently found in specific business problems.

b Substitute the specific known variables shown in Fig. 5 into this formula:

$$S_n = \$500 \cdot \frac{(1.04)^{10}-1}{0.04}.$$

c From Table LXXXIX:

$$S_n = \$500 \cdot \frac{1.48024-1}{0.04}.$$

d Usually in this step a calculating machine should be used if available:

$$S_n = (\$500)\ (12.00611) = \$6{,}003.06,$$

the amount Mr. Smith should have to his credit just after the tenth deposit.

This method is easier than the use of logarithms but more difficult than solution by Table XCI. If the interest rate is found in Table LXXXIX but the number of periods is greater than those given in the tables, this method should be used. The reader will readily see how to use Table LXXXIX to solve problems with periods greater than those given in the table if he remembers the rules of exponents (pp. 141ff., 211ff.).

Solution either by Logarithms or a Principle of Exponents and Calculating Machine

If the rate of interest paid to Mr. Smith by the trust company is assumed to be $3\frac{3}{4}\%$ rather than 4%, then the tables in PRACTICAL MATHEMATICS cannot be used directly for solving the problem. However, the problem may be solved either by the use of logarithms, a principle of exponents and calculating machine, or by interpolation from Table LXXXIX or Table XCI.

Translate the word problem into standard algebraic symbols:

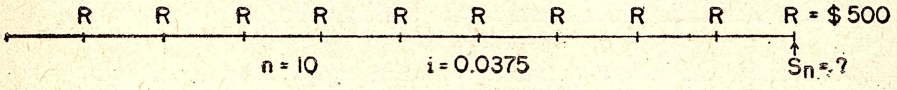


Fig. 6

a Select a formula which contains the independent-variable symbols:

$$S_n = R \cdot \frac{(1+i)^n-1}{i}.$$

b Substitute the known variables shown in Fig. 6 into this formula:

$$S_n = \$500 \cdot \frac{(1.0375)^{10}-1}{0.0375}.$$

c The actual use of logarithms and use of principles of exponents for solving the problem will be left for the reader.

Solution by Interpolation from Table XCI

By employing straight-line interpolation, we may use tables LXXXIX and XCI with a fair degree of accuracy. The use of straight-line interpolation introduces an error because amounts of annuities and compound amounts are curvilinear variables. Furthermore, the higher

the rate of interest, the greater the rate of increase of the slope of the curve for any given time. (Although either Table LXXXIX or XCI can be used as previously indicated, the solution by interpolation will be shown only with Table XCI.)

a Select a formula which contains the table symbols:

$$S_n = R\, s_{\overline{n}|i}.$$

b Substitute the known variables shown in Fig. 6 into the formula:

$$S_n = \$500\, s_{\overline{10}|0.0375}.$$

c Since $3\frac{3}{4}\%$ is between $3\frac{1}{2}\%$ and 4%, then find in Table XCI the following:

$$\begin{aligned} s_{\overline{10}|0.04} &= \$12.00611 \\ s_{\overline{10}|0.035} &= \$11.73139 \\ &\overline{\$\ 0.27472,} \end{aligned}$$

the difference in the amounts to which \$1 per period will accumulate in 10 periods at 4% and at $3\frac{1}{2}\%$.

Since $0.04 - 0.035 = 0.005$

and $0.0375 - 0.035 = 0.0025,$

$$\frac{0.0025}{0.005} \times (\$0.27472) = \$0.13736.$$

This amount should be added to \$11.73139 to give the approximate future value to which \$1 per period for 10 periods would accumulate at $3\frac{3}{4}\%$. That is,

$$\$11.73139 + \$0.13736 = \$11.86875 = s_{\overline{10}|0.0375}, \text{ approximately.}$$

Now, substitute this value in the equation in b:

$$S_n = (\$500)\ (11.86875).$$

d Perform the multiplication by calculating machine, if one is available:

$$S_n = \$5{,}934.38,$$

the amount Mr. Smith should have to his credit just after the tenth deposit.

A comparison of this answer with that of your solution by exponents will indicate that the answer from interpolation is greater by an amount of \$0.46. This variation is due to the fact that the slope of the curve of accumulation of an annuity at 4% is increasing at a greater rate than the slope of the curve of accumulation of an annuity at $3\frac{1}{2}\%$, as was explained in the introduction to this method.

TEST YOUR ABILITY TO SOLVE FOR THE FUTURE VALUE OF CASE 1 ANNUITIES, RENT PAID AT END OF PERIOD

1 A lot was purchased for \$1,500 and taxes in an amount of \$7.50 paid every 3 months for 10 years, the first taxes paid 3 months after the lot was purchased. What was the accumulated value of the lot 10 years after purchase, assuming money worth 4%, converted quarterly?

2 When his daughter was 12 years old, a man decided to place \$240 every 6 months in a building-and-loan association paying 3% converted semi-

annually until she entered college at 17 years of age. How much was available to the girl when she entered college?

3 On January 1, 1944, a man decided to place \$100 per month, beginning January 31, 1944, for the next 20 years in a trust fund paying $3\frac{1}{2}\%$ converted monthly. How much should be credited to his account January 1, 1964?

Amount unknown: present value—According to the terms of a will, Mr. Smith is to receive \$500 semi-annually for the next 5 years from a trust fund. He has an opportunity to sell his rights to a private investor who is willing to purchase these if he can earn 7% semi-annually on his investment. What should Mr. Smith obtain for his rights now?

Translate the word problem into standard algebraic symbols:

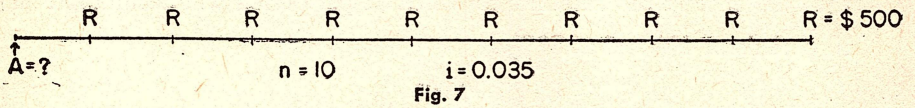


Fig. 7

Solution directly by Table XCII

a Select a formula which contains the table symbols:

$$A = R\, a_{\overline{n}|i}.$$

b Substitute the specific known variables as shown in Fig. 7 into a.

$$A = \$500\, a_{\overline{10}|0.035}.$$

c From Table XCII: $A = (\$500)\ (8.31661)$.

d Complete the calculation: $A = \$4{,}158.31$,
the amount Mr. Smith should receive for his rights now.

Solution by Table XC

a Select a formula which contains the independent-variable symbols:

$$A = R \cdot \frac{1 - v_i^n}{i} \text{ or } R \cdot \frac{1 - (1+i)^{-n}}{i} \text{ or } R \cdot \frac{1 - \frac{1}{(1+i)^n}}{i}.$$

b Substitute the specific known variables as shown in Fig. 7 into a.

$$A = \$500 \cdot \frac{1 - v_{0.035}^{10}}{0.035}.$$

c From Table XC:

$$A = \$500 \cdot \frac{1 - 0.70892}{0.035}.$$

d Complete the calculations:

$$A = \$500 \cdot \frac{0.29108}{0.035} = (\$500)\ (8.31657) = \$4{,}158.29,$$

the amount Mr. Smith should receive for his rights now. (The difference of two cents between this and the preceding answer is due to the use of five-place tables, whereas the answer is in six places.)

Solution by Logarithms

If the rate of interest which the private investor is willing to take is $6\frac{1}{2}\%$ converted semi-annually rather than 7% converted semi-

annually, then the tables in PRACTICAL MATHEMATICS cannot be used directly for solving the problem. However, the problem may be solved by the use of logarithms, a principle of exponents and calculating machine, or by interpolation from Table XC or Table XCII.

Translate the word problem into standard algebraic symbols:

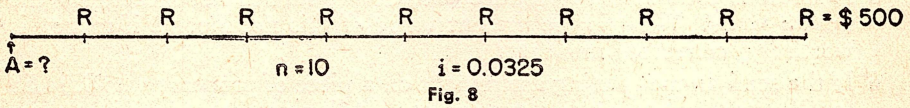


Fig. 8

a Select a formula which contains the independent-variable symbols:

$$A = R \cdot \frac{1 - \frac{1}{(1+i)^n}}{i}.$$

b Substitute the known variables shown in Fig. 8 into the formula:

$$A = \$500 \cdot \frac{1 - \frac{1}{(1.0325)^{10}}}{0.0325}.$$

c The details of solution by logarithms are left to the reader.

Solution by a Principle of Exponents and Calculating Machine

If the exponent is not too large and a calculating machine is available, the principles of exponents may be used rather than employing logarithms. The detail of this solution will be left to the reader. (Cf. p. 1097.)

Solution by Interpolation

If the rate of interest which the private investor is willing to take is $6\frac{1}{2}\%$ converted semi-annually rather than 7% converted semi-annually, then, as has been explained, neither Table XC nor Table XCII can be used directly for solving the problem. By employing straight-line interpolation, however, we may use these tables to get an approximate answer. (Although either Table XC or Table XCII could be used, the solution by interpolation will be shown only with Table XCII.)

a Select a formula which contains the table symbols:

$$A = R\, a_{\overline{n}|i}.$$

b Substitute the known variables shown in Fig. 8 into this formula:

$$A = \$500\, a_{\overline{10}|0.0325}.$$

c Interpolate from Table XCII. Since $3\frac{1}{4}\%$ is between 3% and $3\frac{1}{2}\%$, find in Table XCII the following:

$$a_{\overline{10}|0.03} = \$8.53020$$
$$a_{\overline{10}|0.035} = \$8.31661$$
$$\$0.21359,$$

the difference between the present value of \$1 per period at 0.03 and 0.035 rates of interest. The detailed interpolation will be left to the reader.

TEST YOUR ABILITY TO SOLVE FOR THE PRESENT VALUE OF CASE 1 ANNUITIES, RENT AT END OF PERIOD

4 What is the equivalent cash price, assuming interest at 6% converted monthly, of a house sold on terms like rent at $75.00 per month? The house is to be fully paid for in 8 years.

5 A forest is estimated to produce $10,000 per month net income for the next 3 years. What is the present value of this net income if money is worth 9% converted monthly?

6 If the semi-annual paving assessment on a piece of property was $514.53 for 10 years, what single payment should liquidate the debt just after 5 payments have been made? Assume money worth 5% converted semi-annually.

Rent unknown: future value — What equal annual payment should the XYZ Corporation deposit during the next 10 years to accumulate $1,000,000 in a trust company that pays 4% per annum?

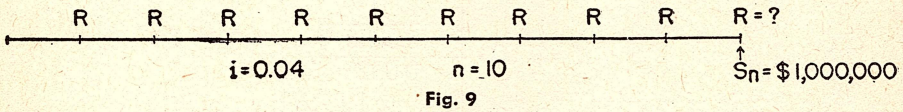


Fig. 9

Solution directly by Table XCIII

It will be noted that Table XCIII shows the values for $\frac{1}{a_{\overline{n}|i}}$, the amortization factor. The values for $\frac{1}{s_{\overline{n}|i}}$, the sinking-fund factor, may be obtained readily from this table by subtracting the appropriate rate of interest. This relationship is shown in the equation,

$$\frac{1}{s_{\overline{n}|i}} = \frac{1}{a_{\overline{n}|i}} - i.$$

If Table XCIII is available and both the interest rate and the number of periods in the specific problem may be found in the table, this method should be used.

a $$R = S_n \cdot \frac{1}{s_{\overline{n}|i}}.$$

b $$R = \$1{,}000{,}000 \cdot \frac{1}{s_{\overline{10}|0.04}}$$

c From Table XCIII:

$$R = (\$1{,}000{,}000)\ (0.08329).$$

d $$R = \$83{,}290,$$

the annual deposit necessary to accumulate a fund of $1,000,000 in 10 years, accurate to five places.

Solution directly by Table LXXXIX

When both the interest rate and the number of periods in the specific problem may be found in Table LXXXIX it may be used. This table

may be used to advantage if the interest rate is found in Table LXXXIX but the number of periods is greater than the tables provide.

a $$R = S_n \cdot \frac{i}{(1+i)^n - 1}.$$

b $$R = \$1{,}000{,}000 \cdot \frac{0.04}{(1.04)^{10} - 1}.$$

c From Table LXXXIX:

$$R = \frac{\$40{,}000}{1.48024 - 1}.$$

d $$R = \$83{,}291.69,$$

the annual deposit necessary to accumulate a fund of $1,000,000 in 10 years. The $1.69 variation between the two methods is due to the use of 5-place table values.

Solution by Logarithms

If the rate of interest paid by the trust company is assumed to be $3\frac{3}{4}\%$ rather than 4%, then the tables in this issue of PRACTICAL MATHEMATICS cannot be used directly for solving the problem. However, the problem feasibly may be solved either by the use of logarithms[1] or by interpolation from Table LXXXIX or Table XCIII.

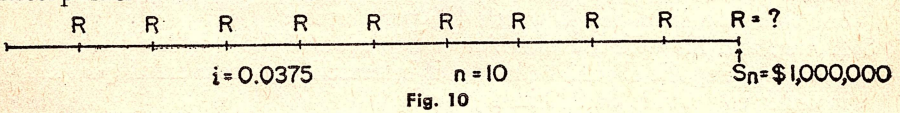


Fig. 10

a $$R = \frac{S_n \cdot i}{(1+i)^n - 1}.$$

b $$R = \frac{(\$1{,}000{,}000)\ (0.0375)}{(1.0375)^{10} - 1}.$$

c The actual solution by logarithms is left to the ingenuity of the reader.

Solution by Interpolation from Table XCIII

a $$R = S_n \cdot \frac{1}{s_{\overline{n}|i}}.$$

b $$R = \$1{,}000{,}000 \cdot \frac{1}{s_{\overline{10}|0.0375}}.$$

c Interpolation from Table XCIII. Since $3\frac{3}{4}\%$ is between $3\frac{1}{2}\%$ and 4%, find in Table XCIII the following:

$$\frac{1}{s_{\overline{10}|0.035}} = \$0.08524$$

$$\frac{1}{s_{\overline{10}|0.040}} = 0.08329$$

$$\$0.00195,$$

the difference between the sinking-fund factor for 10 years at $3\frac{1}{2}\%$ and 4% rates of interest.

[1] The principle of exponents also may be used as explained on p. 901.

Since $0.04-0.035=0.005$ and $0.04-0.0375=0.0025$,

$$\frac{0.0025}{0.005}\cdot(\$0.00195)$$

should be added to \$0.08329 to give the approximate amount that will be required as a deposit each year to accumulate a sinking fund of \$1 at the end of 10 years when interest is 0.0375 effective:

$$\$0.08329+\$0.000975=\$0.084265=\frac{1}{s_{\overline{10}|0.0375}}, \text{ approximately.}$$

d By substituting this value, we get

$$R=(\$1{,}000{,}000)\ (0.084265).$$

e

$$R=\$84{,}265,$$

approximately the annual deposit necessary to accumulate a fund of \$1,000,000 in 10 years.

TEST YOUR ABILITY TO SOLVE FOR RENT OF FUTURE VALUE OF CASE 1 ANNUITIES, RENT PAID AT END OF PERIOD

7 The treasurer of a senior class of a college sold his 500 classmates the idea of contributing annually sufficient money to enable them to give \$35,000 to their alma mater at the time of their tenth reunion. How much would each member of the class have to donate each year to carry out this plan if money is worth 4% per annum?

8 A man and his wife are planning a trip around the world. What equal quarterly deposit should be made, if interest is 4% converted quarterly, in order to provide \$4,000 in 4 years?

9 A man decided to accumulate a fund of \$25,000 in 10 years. How much would he have to save monthly in order to develop this fund if money is worth $4\frac{1}{2}\%$ converted monthly?

Rent unknown: present value—What equal monthly payment is necessary to liquidate a debt of \$500 in 15 months if interest is 2% per month?

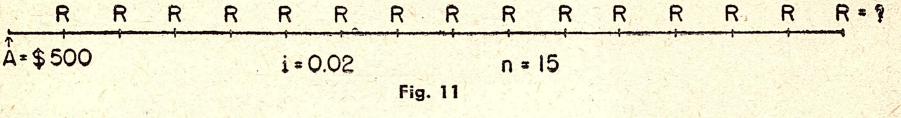


Fig. 11

Solution directly by Table XCIII

a

$$R=\frac{1}{a_{\overline{n}|i}}.$$

b

$$R=\$500\cdot\frac{1}{a_{\overline{15}|0.02}}.$$

c From Table XCIII:

$$R=(\$500)\ (0.07783).$$

d

$$R=\$38.91,$$

the monthly payment necessary to liquidate a debt of \$500 in 15 months, both principal and interest at 2% per month on the unpaid balance.

Solution directly by Table XC

a $$R = A \cdot \frac{i}{1 - v_i^n} = \frac{A\,i}{1 - v_i^n}.$$

b $$R = \frac{(\$500)\ (0.02)}{1 - v_{0.02}^{15}}.$$

c From Table XC: $$R = \frac{\$10}{1 - 0.74302}.$$

d By use of calculating machine or logarithms:

$$R = \frac{\$10}{0.25698} = \$38.91,$$

the annual payment necessary to liquidate a debt of \$500 in 15 months, both principal and interest at 2% per month on the unpaid balance.

If both the interest rate and the number of periods in the specific problem may be found in Table XC, this method may be used. This table may be used to advantage if the interest rate is found in Table XC but the number of periods is greater than the tables provide.

Solution by Logarithms

If the rate of interest earned by the debt is assumed to be $2\frac{1}{8}\%$ instead of 2%, the tables on pages 000 to 000 cannot be used directly for solving the problem. However, the problem feasibly may be solved either by the use of logarithms or by interpolation from Table XC or Table XCIII. (The solution of the problem with the new assumption and by these two methods is left as an exercise for the reader.)

TEST YOUR ABILITY TO SOLVE FOR THE UNKNOWN RENT IN PRESENT VALUE, CASE 1 ANNUITIES, RENT PAID AT END OF PERIOD

10 The purchaser of a \$4,000 house arranged to pay the debt principal and interest at 5% converted quarterly, in 40 equal quarterly payments. After making 20 payments, how much does he still owe?

11 Three sons whose ages are 9, 12, and 14 are left \$50,000 in a will. At the age of 21, each is to receive an equal 10-year semi-annual annuity. If money is worth $3\frac{1}{2}\%$ converted semi-annually, what semi-annual rent will each receive?

12 A man who has the right to receive \$5,000 in 7 years wishes to sell his right now. A financier offers to pay him \$1,000 cash and a periodic semi-annual payment so as to yield 7% semi-annually. What should be the semi-annual payment?

Number of periods unknown: future value—How many payments of \$100 per month will it take to accumulate \$4,000 if money is worth 6% converted monthly?

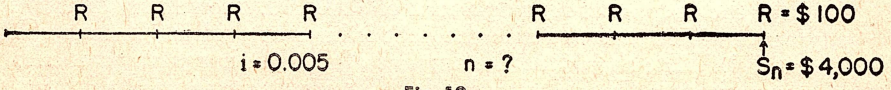


Fig. 12

Solution by Table XCI

a $s_{\overline{n}|i}=\frac{S}{R}$.

b $s_{\overline{n}|0.005}=\frac{4{,}000}{100}=\$40.00$.

c From Table XCI, find in $\frac{1}{2}\%$ column:

$$s_{\overline{36}|0.005}=\$39.33611;\ s_{\overline{37}|0.005}=\$40.53279.$$

d $n=36+$ months,

which is accurate enough for practical purposes because it is customary practice to arrange for either a larger thirty-sixth payment or a smaller thirty-seventh payment rather than to provide for the payment of a fractional part of the regular payment at the end of a fractional part of the last period.

Solution by Table LXXXIX

a $(1+i)^n=\frac{S_n \cdot i}{R}+1$.

b $(1.005)^n=\frac{\$4{,}000(0.005)}{100}+1=\$1.200$,

the amount that should be accumulated from a \$1 deposit at compound interest for n periods at 5%.

c From Table LXXXIX find in the $\frac{1}{2}\%$ column:

$$(1.005)^{36}=\$1.19668;\ (1.005)^{37}=\$1.20266.$$

d Again, $n=36+$.

If the value of n is beyond the limits of the table, divide the table limit value for a given rate of interest into the desired sum—that is

$$\frac{(1+i)^n}{\text{value in table limit}}.$$

The result of this division may then be used for $(1+i)^n$. To the n derived from this solution should be added n corresponding to the limit of the table in order to determine the required number of payments.

Solution by Logarithms

Solution by logarithms may be preferred to table solutions when solving for the number of periods. In addition, when the rate of interest is not found in the table, solution by logarithms is definitely better than solution by interpolation from tables.

TEST YOUR ABILITY TO SOLVE FOR THE UNKNOWN NUMBER OF PERIODS IN FUTURE VALUE CASE 1 ANNUITIES, RENT PAID AT END OF PERIOD

13 A corporation estimates that it can place \$5,000 quarterly into a sinking fund for the purpose of retiring a loan of \$200,000. How many quarterly payments will it take to retire the debt if interest paid is 3% converted quarterly? What should be the amount of the last irregular payment?

14 A man estimates that he can deposit \$150 per quarter in a fund paying

2% converted quarterly. How many years will it take him to accumulate \$5,000? Assuming the last payment to be a regular rather than a partial payment, how much should be credited to his account just after the last payment?

Number of periods unknown: present value—How many annual payments of \$600 will be required to pay for a \$5,000 house if interest is 5% per annum?

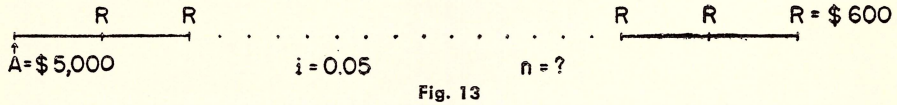


Fig. 13

Solution by Table XCII

a $a_{\overline{n}|i} = \dfrac{A}{R}$

b $a_{\overline{n}|0.05} = \dfrac{\$5{,}000}{600} = \$8.33333.$

c From Table XCII find in the 5% column: $a_{\overline{12}|0.05} = \$8.86325$, $a_{\overline{11}|0.05} = \$8.30641$.

d $n = 11+.$

Solution by Table XC

a $v_i^n = 1 - \dfrac{Ai}{R}.$

b $v_{0.05}^n = 1 - \dfrac{(\$5{,}000)\ (0.05)}{600} = \$1 - \$0.41667 = \$0.58333.$

c From Table XC find in the 5% column: $v_{0.05}^{11} = \$0.58468$; $v_{0.05}^{12} = \$0.55684$.

d $n = 11+.$

Solution by Logarithms

Solution by logarithms may be preferred to table solutions when solving for the number of periods. In addition, if the rate of interest is $4\frac{3}{4}\%$ rather than 5%, the logarithmic solution is definitely better than solution by interpolation from tables. Therefore, even though this problem may be solved by interpolation from Tables XC and XCII, only the solution by logarithms will be illustrated.

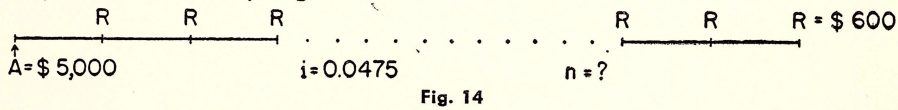


Fig. 14

a $$n = \frac{\log 1 - \log\left(1 - \dfrac{Ai}{R}\right)}{\log(1+i)}.$$

b By substitution:

$$n = \frac{0 - \log\left[1 - \dfrac{(5{,}000)\ (0.0475)}{600}\right]}{\log 1.0475} = \frac{0 - \log 0.604167}{\log 1.0475}.$$

c From logarithmic tables:

$$\log 0.60417 = \bar{1}.781157$$
$$\log 1.0475 = 0.020154.$$
$$\log 1 = 1.000000 - 1$$
$$-\log 0.604167 = -0.781157 - 1$$
$$\log 1 - \log 0.604167 = \overline{0.218843}.$$

By substitution,

$$n = \frac{0.218843}{0.020154} = 10.86 \text{ years.}$$

TEST YOUR ABILITY TO SOLVE FOR THE UNKNOWN NUMBER OF PERIODS IN PRESENT VALUE CASE 1 ANNUITIES, RENT PAID AT END OF PERIOD

15 An automobile manufacturer desires to establish a plan by which he can sell his cars at $25.00 per month. If he must receive $1\frac{1}{2}\%$ per month to cover costs of operating the installment plan, how many monthly payments will be required to pay for a $1,000 automobile? What should be the amount of the last payment if the unpaid balance is due at the time of the last regular payment?

16 How many months will it take to repay a loan of $300 if interest is $3\frac{1}{2}\%$ per month and payments are $20 per month?

17 How many weeks will it take to liquidate a debt of $100 if payments are $5 per week and the rate of interest is 40% converted weekly?

18 The benefits of a matured endowment policy call for $10,000 cash or $1,000 cash and an annuity certain of $100 per month until the balance is exhausted. If the unpaid balance is credited with interest at the rate of 3% per annum converted monthly, how long will the monthly payments continue if the annuity option is selected?

Rate of interest unknown: future value—What rate of interest would have to be earned for a trustee to accumulate a fund of $15,000 in 10 years from an annual income of $1,250 per year?

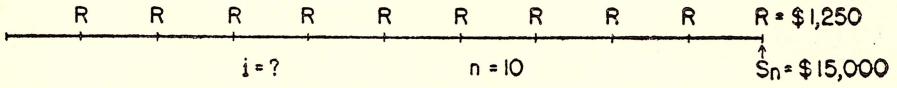


Fig. 15

Solution by Table XCI

a $$s_{\overline{n}|i} = \frac{S_n}{R}.$$

b $$s_{\overline{10}|i} = \frac{\$15{,}000}{\$1{,}250} = \$12.00000.$$

c From Table XCI find in row $n = 10$:

$s_{\overline{10}\|0.04}$	$12.00611	$s_{\overline{10}\|i}$ = $12.00000	0.04
$s_{\overline{10}\|0.035}$	11.73139	$s_{\overline{10}\|0.035}$ = 11.73139	0.035
	$ 0.27472	$ 0.26861	0.005

d $$i=0.035+\left(0.005\cdot\frac{0.26861}{0.27472}\right)=0.03989,$$

approximate rate of interest.

This may be proved by the logarithmic solution for the future value of an annuity.

Solution by Table LXXXIX

The solution by Table LXXXIX will not be illustrated since it follows a pattern of interpolation between interest rates like the preceding. In addition, the use of Table LXXXIX for locating the variables, i and n, is quite similar.

TEST YOUR ABILITY TO SOLVE FOR THE UNKNOWN INTEREST IN FUTURE VALUE CASE 1 ANNUITIES, RENT PAID AT END OF PERIOD

19 An insurance company advertises, "Save now, retire early". A premium of \$500 per year, the first payment at age 35, will produce a fund at age 65 of \$21,189.72. What rate of interest would be earned under this contract?

20 If 40 quarterly payments of \$100 will amount to \$4,566.75, what is the rate of interest?

21 What is the annual rate of interest if \$200 per year will amount to \$9,515.08 in 30 years?

Rate of interest unknown: present value — A will provides for \$1,000 per year for the next 10 years. If the beneficiary is offered \$8,200 for his rights, what is the rate of interest?

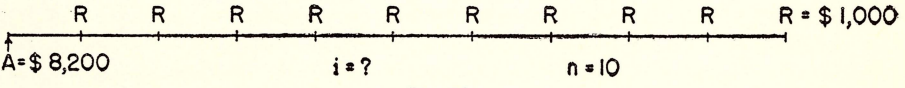


Fig. 16

Solution by Table XCII

a $$a_{\overline{n}|i}=\frac{A}{R}.$$

b $$a_{\overline{10}|i}=\frac{\$8,200}{1,000}=\$8.20.$$

c From Table XCII find in row $n=10$:

$a_{\overline{10}\|0.035}=\$8.31661$	$a_{\overline{10}\|0.035}=\$8.31661$	0.04
$a_{\overline{10}\|0.04}=\ 8.11090$	$a_{\overline{10}\|i}=\ 8.20000$	0.035
Difference $=\$0.20571$	Difference $=\$0.11661$	0.005

d $$i=0.035+0.005\cdot\frac{0.11661}{0.20571}=0.03783,$$

approximate rate of interest in the contract.

Solution by Table XC

The solution by Table XC will not be illustrated since it follows a pattern of interpolation between interest rates like the preceding. Furthermore, the use of Table XC for locating the variables, i and n, is quite similar.

TEST YOUR ABILITY TO SOLVE FOR THE UNKNOWN RATE OF INTEREST IN PRESENT VALUE, CASE 1 ANNUITIES, RENT PAID AT END OF PERIOD

22 A loan company offers to lend \$100 on your furniture or automobile. Repayment is to be made in 6 monthly payments of \$18.50, the first coming due one month from now. What rate of interest per month is being charged?

23 A building-and-loan association advertises as follows, "We will build you any type of house you desire . . . . You pay 1 per cent of the cost every month for the next 12 years and the house is yours." Find the approximate interest rate charged.

24 The so-called 6% time-payment plan used by many banks and finance companies is as follows: If the unpaid balance on a new car or other durable consumer goods is \$500, 6% is added. Then the purchaser pays \$44.17 per month or one-twelfth of \$530. What is the actual rate, convertible monthly?

CASE 2. INSTALLMENT-PAYMENT PERIOD AND INTEREST-CONVERSION PERIOD ARE NOT EQUAL

As has been noted previously, there is a group of contracts in which the interest-conversion period and the rent period are not equal. It is apparent, however, that, if the relationship between the interest-conversion period and the rent period in terms of compound interest is cast into a mathematical equation, an equation developed for the solution of Case 1 problems may be mathematically manipulated to produce an equation for the solution of Case 2 problems.

If m equals the number of months in the rent period divided by the number of months in the interest-conversion period, and r equals the rate of interest per rent period, it may be shown readily that

$$(1+i)^m-1=r \text{ or } (1+i)^m=1+r.$$

In Case 1 formulas, i has been the interest rate per rent period as well as the interest rate per interest-conversion period. Now, the two basic formulas for Case 1 will be restated with r instead of i so as to differentiate the two interest rates.

$$S_n=R\cdot\frac{(1+r)^n-1}{r}.$$

$$A=R\cdot\frac{1-(1+r)^{-n}}{r}.$$

Now, substituting in Case 1 formulas the interest rate per rent period in terms of the interest rate per conversion period, we get:

$$S_{mn}=R\cdot\frac{(1+i)^{mn}-1}{(1+i)^m-1}=R\cdot\frac{(1+i)^{mn}-1}{i}\cdot\frac{i}{(1+i)^m-1}=R\cdot s_{\overline{mn}|i}\cdot\frac{1}{s_{\overline{m}|i}}.$$

$$A_{mn}=R\cdot\frac{1-(1+i)^{-mn}}{(1+i)^m-1}=R\cdot\frac{1-(1+i)^{-mn}}{i}\cdot\frac{i}{(1+i)^m-1}=R\cdot a_{\overline{mn}|i}\cdot\frac{1}{s_{\overline{m}|i}}.$$

It is apparent that a detailed analysis similar to Case 1 could now be carried out, solving for each of the five variables as an unknown. However, since the complete pattern of analysis has already been

demonstrated, application to one future-value and one present-value problem should be sufficient to enable the reader to utilize effectively these two formulas for solving the great variety of financial problems to which they are applicable. One example will be used to demonstrate the mathematics in a contract in which the interest-conversion period is longer than the rent period. The other will show the simpler mathematical relations in which the interest-conversion period is shorter than the rent period. One example will demonstrate the solution for the unknown rent; the other will show a solution for the amount of an annuity. One solution requires the present-value formula, the other the future-value formula.

Amount unknown: future value: interest-conversion period longer than rent period—A man decides to deposit \$100 per month in a building-and-loan association which pays 3% converted quarterly. How much should he have on deposit at the end of 3 years?

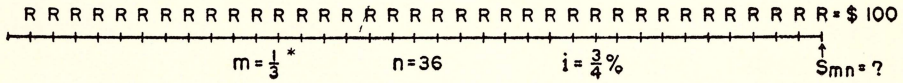


Fig. 17

a
$$S_{mn}=R\cdot\frac{(1+i)^{mn}-1}{(1+i)^{m}-1}.$$

b
$$S_{mn}=\$100\cdot\frac{(1.0075)^{12}-1}{(1.0075)^{\frac{1}{3}}-1}.$$

c From Tables XCIV and LXXXIX:
$$S_{mn}=\$100\cdot\frac{0.09381}{0.00249}=\$3{,}767.46.$$

Rent unknown: present value: interest-conversion period shorter than rent period—A farmer desires to liquidate a \$10,000 loan in 10 equal annual payments. If the interest is 5% converted quarterly, what should the annual payment be?

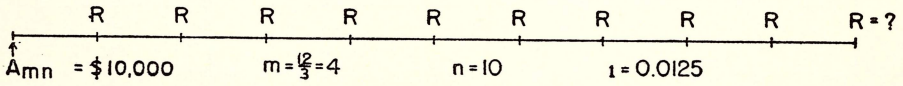


Fig. 18

a
$$A_{mn}=R\cdot a_{\overline{mn}|i}\cdot\frac{1}{s_{\overline{m}|i}}.$$

$$R=A_{mn}\cdot s_{\overline{m}|i}\cdot\frac{1}{a_{\overline{mn}|i}}.$$

* Fractional values for m are given in Table XCIV. However, values for $s_{\overline{m}|i}$ when m is a fraction are not given in tabular form.

b $R = \$10{,}000\, s_{\overline{4}|0.0125} \cdot \dfrac{1}{a_{\overline{40}|0.0125}}$.

c $R = \$10{,}000 \cdot 4.07563 \cdot 0.03192 = \$1{,}300.94$.

TEST YOUR ABILITY TO SOLVE CASE 2 ANNUITIES, RENT PAID AT END OF PERIOD

25 A man desires to pay off a debt of $3,000 with interest at 5% per annum in 24 equal monthly payments. What should be the monthly payment?

26 A television set is purchased for $400 with a down payment of $100, the unpaid balance and finance charge to be paid in 12 equal monthly payments. To determine the monthly payment, add 6% of $400 to the $300 as a finance charge and divide the result by 12. What effective rate of interest is charged?

27 A young man is planning to accumulate an estate. He estimates that he can save $50 a quarter for the first 5 years, $75 a quarter for the second 5 years, and $100 a quarter for the next 10 years. If he can obtain 3% converted semi-annually, how much should he have to his credit at the end of this 20-year period?

28 A father calculates that he can place $400 a year in a fund to provide his boy $100 per month for 4 years of college, the first $100 payment to be made one month after the last $400 payment. If the investment can be made to yield $3\frac{1}{2}\%$ converted semi-annually, how many years should the $400 payment be made?

29 A man bought a house for $5,000. He agreed to pay off this loan principal and interest with $150 quarterly payments. After paying for 10 quarters, he wishes to know how much he still owes on the principal. Assume interest at 5% converted semi-annually.

30 A professional man aged 30 was projecting a savings plan to age 60. He assumed that he could save twice as much per year for the second period of five years as for the first period of five years, half again as much per year for the third period of five years as for the second period of five years, fifty per cent more per year for the fourth period of five years than for the third period of five years, and twenty-five per cent more per year for the fifth and final period of ten years than for the fourth period of five years. How much per annum should he deposit in each of these years in order to have $25,000 when 60 years of age if a trust company will pay him 3% per annum converted semi-annually?

Installment payments at beginning of period—annuity due

Many important installment contracts provide for the periodic payment to be made at the beginning rather than at the end of the period. It easily may be observed that in the annuity-due contracts each periodic payment is discounted one less period in present value and earns interest for one more period in future-value contracts than in ordinary annuity. This indicates that there is a simple relationship between the mathematical formulas for solving annuity due and those for solving ordinary annuity problems.

CASE 1. INSTALLMENT-PAYMENT PERIOD AND INTEREST-CONVERSION PERIOD COINCIDE

The general time chart for the future value of Case 1 ordinary annuity is shown in Fig. 19. The general time chart for the future value of an annuity due is shown in Fig. 20. It now may be observed that, if a payment of R is assumed to be made at the time of the future valuation, the time chart would be represented as in Fig. 21. It is now apparent that the formula for the future value of an ordinary annuity for one additional period could be used to solve problems of this type if the assumed final rent actually were made. The formula then would be

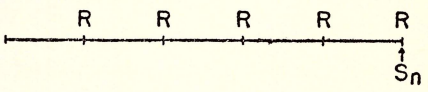


Fig. 19

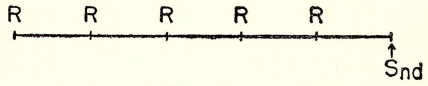


Fig. 20

$$S_n = Rs_{\overline{n+1}|i} = R \cdot \frac{(1+i)^{n+1}-1}{i}.$$

The last R is not actually paid; therefore, it should be subtracted from the right-hand member of the equation in order for the equation to fit the situation of an annuity due. The formula then becomes:

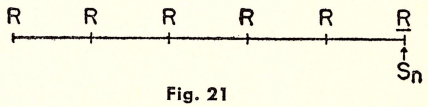


Fig. 21

$$S_{nd} = R\,(s_{\overline{n+1}|i} - 1) = R\left[\frac{(1+i)^{n+1}-1}{i} - 1\right].$$

Similarly, the time chart may be used to visualize present-value annuity-due formulas in terms of ordinary-annuity formulas. The general time chart for the present value of an ordinary annuity is given in Fig. 22. The general time chart for the present value of an annuity due is presented in Fig. 23. It may be observed that, if the first payment of R is assumed not to be made, the time chart would take the form shown in Fig. 24. It is now apparent that the formula for the present value of an ordinary annuity for one less period could be used to solve problems of this type if the first payment actually were not made. The formula then would be

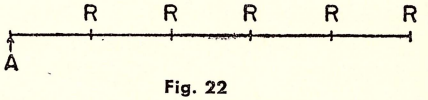


Fig. 22

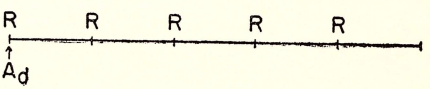


Fig. 23

$$A_d = Ra_{\overline{n-1}|i} = R \cdot \frac{1-(1+i)^{-(n-1)}}{i}.$$

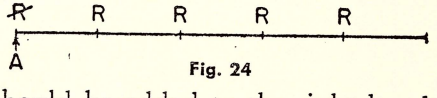


Fig. 24

Since the first R is actually paid, it should be added to the right-hand member of the equation in order to fit the situation of an annuity due. The formula then becomes:

$$A_d = R(a_{\overline{n-1}|i} + 1) = R\left[\frac{1-(1+i)^{-(n-1)}}{i} + 1\right].$$

A complete analysis of the types of problems which may be solved with each of the formulas, present and future values annuity due, would demand specific application to problems requiring respectively solutions for each of the four variables or eight types of problems. Again, however, rather than elaborate the explanation, we shall merely direct attention to the complete pattern of analysis shown under Case 1 ordinary annuities and present two illustrative examples.

Rent unknown: future value: rent paid at beginning of period—What semi-annual payment beginning now will be required to accumulate a fund of \$10,000 in 5 years if interest is 4% converted semi-annually?

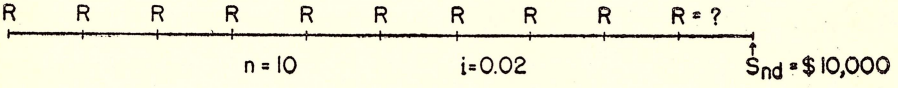


Fig. 25

a $S_{nd}=R(s_{\overline{n+1}|i}-1)$.

b $\$10{,}000=R(s_{\overline{11}|0.02}-\$1)$.

c From Table XCI:

$$\$10{,}000=R(\$12.16872-\$1)=R(\$11.16872).$$

d $$R=\frac{\$10{,}000}{\$11.16872}=\$895.36.$$

Number of periods unknown: present value: rent paid at beginning of period—How long will it take to pay for an automobile valued at \$1,000 if the rate of interest is 2% per month on the unpaid balance and the monthly payment is \$50, the first payment due at time of purchase?

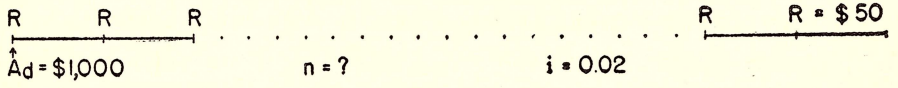


Fig. 26

a $A_d=R(a_{\overline{n-1}|i}+1)$.

b $\$1{,}000=\$50\ (a_{\overline{n-1}|0.02}+\$1)$.

c From Table XCII:

$$a_{\overline{n-1}|0.02}=\frac{\$1{,}000}{50}-\$1=\$19.00.$$

d $n-1=24$ with a larger than regular final payment or 25 with a smaller than regular final payment.

$n=25$ or 26 months.

TEST YOUR ABILITY TO SOLVE CASE 1 ANNUITIES, RENT PAID AT BEGINNING OF PERIOD

31 A real estate company wishes to know the amount of the quarterly payment, the first payment to be made at time of purchase, necessary to pay for a \$5,000 house if interest is 5% converted quarterly and the house is to be paid for in 10 years.

32 A father wishes to accumulate a fund for his daughter's education to be available at the time she is 18 years of age. He deposited \$300 in a trust fund paying 3% converted semi-annually on the day of her birth and each 6 months thereafter until she was $17\frac{1}{2}$ years of age. What should be credited to the daughter's account upon her eighteenth birthday?

33 The option of an insurance policy provides for \$5,000 in cash or equal annual payments for 15 years, the first payment due at once. What should be the annual payment if interest is allowed at 3% per annum?

34 An electrical appliance is sold on terms of \$10 a month for 24 months, the first \$10 due at time of purchase. What is the equivalent cash price if interest is computed at the rate of $2\frac{1}{2}\%$ per month?

35 How much is still owed on the appliance in problem 34 just before the tenth payment is made?

36 Compare (1.04) $s_{\overline{20}|0.04}$ with $s_{\overline{21}|0.04}-1$. How do you explain this relationship?

37 (a) How many payments will have to made to pay for a \$10,000 loan if payments of \$250 are made at the beginning of each quarter and interest is 6% converted quarterly? (b) How much will be due just after the fifteenth payment has been made?

38 What is the interest rate in a contract for the purchase of an automobile under the following terms:

Total cost price	\$457.13
Total down payment	187.00
Deferred balance	\$270.13
Time sales mark-up	147.23
Time balance	\$438.36

The debt is to be liquidated in 12 equal monthly payments of \$36.53.

CASE 2. INSTALLMENT-PAYMENT PERIOD AND INTEREST-CONVERSION PERIOD ARE NOT EQUAL

Utilizing the procedure for converting Case 1 formulas to Case 2 formulas previously outlined, of substituting the interest rate per rent period in terms of the rate per conversion period, we re-state the annuity-due formulas of Case 1 as:

FUTURE VALUE

$$S_{nd}=R\left[\frac{(1+i)^{m(n+1)}-1}{(1+i)^m-1}-1\right]=R\cdot\frac{(1+i)^{mn+m}-1-(1+i)^m+1}{(1+i)^m-1}$$

$$=R\cdot\frac{(1+i)^m[(1+i)^{mn}-1]}{(1+i)^m[1-(1+i)^{-m}]}=R\cdot\frac{(1+i)^{mn}-1}{i}\cdot\frac{i}{1-(1+i)^{-m}}=R\cdot s_{\overline{mn}|i}\cdot\frac{1}{a_{\overline{m}|i}}.$$

PRESENT VALUE

$$A_d=R\left[\frac{1-\dfrac{1}{(1+i)^{m(n-1)}}}{(1+i)^m-1}+1\right]=R\cdot\frac{(1+i)^m[1-(1+i)^{-mn}]}{(1+i)^m[1-(1+i)^{-m}]}$$

$$=R\cdot\frac{1-(1+i)^{-mn}}{i}\cdot\frac{i}{1-(1+i)^{-m}}=R\cdot a_{\overline{mn}|i}\cdot\frac{1}{a_{\overline{m}|i}}.$$

These two formulas make possible the solution of ten specific types of

problem. This results from the fact that each of the two equations contain five variables which, respectively, may be the unknown in a different kind of financial contract or agreement.

It is apparent that both these formulas, however, result in patterns of related variables which have previously been demonstrated. Therefore, the reader should be able to make effective application of these formulas without any illustrative solutions.

TEST YOUR ABILITY TO SOLVE CASE 2 ANNUITIES, RENT PAID AT BEGINNING OF PERIOD

39 To pay for a loan of \$6,000, principal and interest, C agrees to pay R dollars at the first of each month for 3 years. Assuming a rate of 5% quarterly, find R.

40 An option in a \$5,000 matured endowment policy allows \$100 per month beginning immediately. How many certain payments should be received if interest is 3% converted quarterly?

41 Compute the monthly rent necessary to produce a sinking fund of \$100,000 in three years. Assume an interest rate of 8% converted quarterly, the first deposit to be made immediately.

Installment payments deferred or terminated

Another major group of installment contracts is characterized by a time interval between the present value and the first rent payment which is more than one rent period. This group of contracts is often referred to as deferred annuities. The parallel future-value class of installment contracts is differentiated from previous classes by a time interval between the future value and the last rent payment which is greater than one rent period. This type of periodic payment is here called terminated annuity. As in previous analyses, problems of these two major types will be divided into Cases 1 and 2.

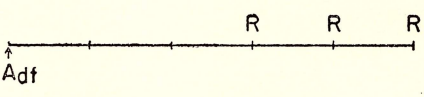


Fig. 27

CASE 1. INSTALLMENT-PAYMENT PERIOD AND INTEREST-CONVERSION PERIOD COINCIDE

A deferred annuity may be charted as in Fig. 27. A deferred annuity may be transformed to an ordinary annuity by assuming that the deferred rents were paid, as in Fig. 28. Let k stand for the number of assumed or unpaid rents. Then an examination of Fig. 28 will disclose that the present value of a deferred annuity may be expressed as the difference between two ordinary annuities. This relationship may be expressed in equation form:

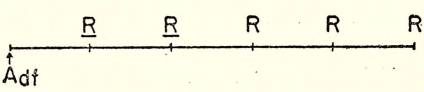


Fig. 28

$$A_{df} = R(a_{\overline{n+k}|i} - a_{\overline{k}|i}) = R\left[\frac{1-(1+i)^{-(n+k)}}{i} - \frac{1-(1+i)^{-k}}{i}\right].$$

A terminated annuity may be charted as in Fig. 29. By assuming that the terminated rents were paid, we may convert the terminated annuity into an ordinary annuity. The chart then would be like Fig. 30. The value of the terminated annuity may be expressed as the difference between two future-value annuities:

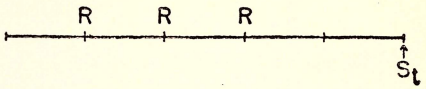


Fig. 29

$$S_t = R(s_{\overline{n+k}|i} - s_{\overline{k}|i}) = R\left[\frac{(1+i)^{n+k}-1}{i} - \frac{(1+i)^{k}-1}{i}\right].$$

At this point, the reader is again called upon to use his imagination to project the pattern of analysis necessary to encompass the ten types of financial problems which may be solved by these two basic equations. These ten types of problems result as each of the five variables in each of the two equations is respectively considered as an unknown. Only one type problem for each of the two formulas will be presented as an introduction to this projected analysis.

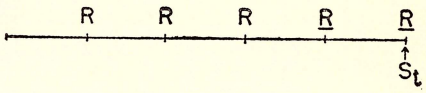


Fig. 30

Amount unknown: deferred annuity: Case 1—How much should be deposited in a fund paying 2% converted quarterly at age 50 to provide 10 quarterly payments of $1,000, the first payment due at age 53?

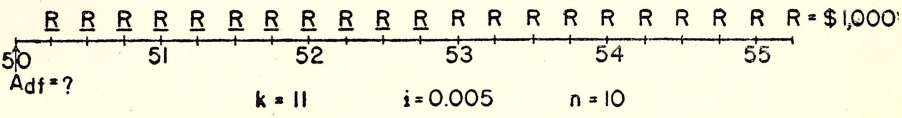


Fig. 31

a $A_{df} = R(a_{\overline{n+k}|i} - a_{\overline{k}|i})$

b $A_{df} = \$1{,}000\ (a_{\overline{21}|0.005} - a_{\overline{11}|0.005})$.

c $A_{df} = \$1{,}000\ (\$19.88798 - \$10.67703) = \$9{,}210.95$.

Amount unknown: terminated annuity: Case 1—How much should be in a fund at the end of 5 years if $500 semi-annually is deposited for the first 3 years, assuming money worth $2\frac{1}{2}\%$ converted semi-annually?

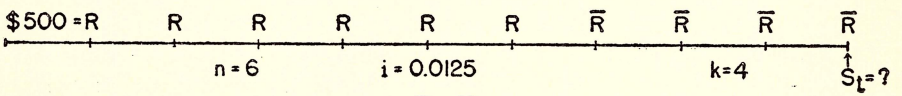


Fig. 32

a $S_t = R(s_{\overline{n+k}|i} - s_{\overline{k}|i})$.

b $S_t = \$500(s_{\overline{10}|0.0125} - s_{\overline{4}|0.0125})$.

c $S_t = \$500\ (\$10.58167 - \$4.07563) = \$3{,}253.02$.

CASE 2. INSTALLMENT-PAYMENT PERIOD AND INTEREST-CONVERSION PERIOD ARE NOT EQUAL

Here, too, the procedure previously introduced of substituting the interest rate per period in terms of the interest rate per conversion period will be used for converting Case 1 to Case 2 formulas. The results are:

DEFERRED ANNUITY—PRESENT VALUE:

$$A_{df} = R\left\{\left[\frac{1-(1+i)^{-m(n+k)}}{(1+i)^m-1}\right]-\left[\frac{1-(1+i)^{-mk}}{(1+i)^m-1}\right]\right\}$$

$$= R\left\{\left[\frac{1-(1+i)^{-m(n+k)}}{i}\right]-\left[\frac{1-(1+i)^{-mk}}{i}\right]\right\}\cdot\frac{i}{(1+i)^m-1}$$

$$= R\,(a_{\overline{m(n+k)}|i}-a_{\overline{mk}|i})\,\frac{1}{s_{\overline{m}|i}}.$$

TERMINATED ANNUITY—FUTURE VALUE:

$$S_t = R\left\{\left[\frac{(1+i)^{m(n+k)}-1}{(1+i)^m-1}\right]-\left[\frac{(1+i)^{mk}-1}{(1+i)^m-1}\right]\right\}$$

$$= R\left\{\left[\frac{(1+i)^{m(n+k)}-1}{i}\right]-\left[\frac{(1+i)^{mk}-1}{i}\right]\right\}\cdot\frac{i}{(1+i)^m-1}$$

$$= R\,(s_{\overline{m(n+k)}|i}-s_{\overline{mk}|i})\,\frac{1}{s_{\overline{m}|i}}.$$

TEST YOUR ABILITY TO SOLVE CASE 1 AND CASE 2 DEFERRED AND TERMINATED ANNUITIES

42 It is estimated that a grove will provide an income of $10,000 per year for 5 years. The first income is assumed to be available 6 years from now. After that, it is estimated that the grove will produce $15,000 per year net on the average for 20 years. If interest is assumed of (a) 5%, (b) 6% effective, what is the present value of the grove?

43 A professional man aged 30 is projecting a savings plan. He believes that he can save according to the following schedule: $500 per year for the first 5 years, the first deposit at age 31; $1,000 per year for the next 5 years; $2,000 per year for the next 10 years; and $2,500 per year for the next 10 years. How much should he have to his credit at age 60 if a trust company will pay him $2\frac{1}{2}\%$ per annum?

44 A machine is worth $10,000. Terms are arranged as follows: $2,000 cash, $200 per month for 2 years, then $300 per month until paid. The final monthly payment is to be a fractional payment. These payments provide for repayment of principal and interest at 12% per annum converted monthly. (a) How many payments will it take to pay for the machine? (b) What should be the amount of the final fractional payment?

45 An all-expense tour was purchased for $750.00 on June 15, with interest, in 10 monthly installments, the first payment due October 15. If we assume interest at 9% converted monthly, what should the monthly payment be?

Perpetual installment payments

Still another group of installment arrangements is characterized by the assumption of an infinite number of payments. It is at once apparent that the future value of an infinite number of installments is meaningless. It will be shown, however, that, mathematically, the present value of an infinite series of payments is converging and therefore can be made to show a finite result. An infinite series of regular periodic payments is called a perpetuity. Equations and illustrations of this class of installment will be limited to Case 1.

DERIVATION OF PERPETUITY FORMULA

The formula for a perpetuity is easily derived from the formula for the present value of an ordinary annuity. Substitute ∞ for n and A_∞ for A in the present value of an ordinary annuity formula; the following results:

$$A_\infty = R \cdot \frac{1-\frac{1}{(1+i)^\infty}}{i} = \frac{R}{i}.$$

Observe that there are three variables, each of which may be unknown in a particular type of problem. Here, however, only one type will be shown.

Present value unknown: perpetuity—What is the amount of a fund necessary to provide a \$40,000 per year endowment for the upkeep of a library, if money is worth 2% per annum?

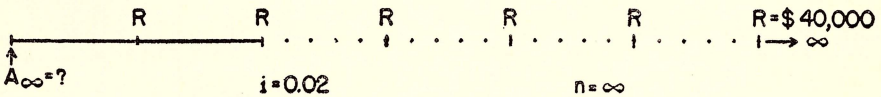


Fig. 33

a $$A_\infty = \frac{R}{i}$$

b $$A_\infty = \frac{\$40{,}000}{0.02}$$

c $$A_\infty = \$2{,}000{,}000.$$

DERIVATION OF CAPITALIZED-COST FORMULA

One of the variations of a perpetuity is referred to as *capitalized cost*, the fund necessary to provide both a present amount with which to purchase an asset and a fund which will produce sufficient income to provide a sinking fund accumulating regularly to a predetermined amount at regular intervals. Usually, this periodic fund is to be used in reconditioning or replacing the original asset.

If C_∞ = capitalized cost

O = original cost of asset

W = periodic replacement cost

k = number of periods between replacements

i = rate of interest earned per period earned by the fund

$W \cdot \frac{1}{s_{\overline{k}|i}}$ = rent necessary to accumulate W every k years in a sinking fund paying i rate of interest,

capitalized cost may be charted as in Fig. 34.

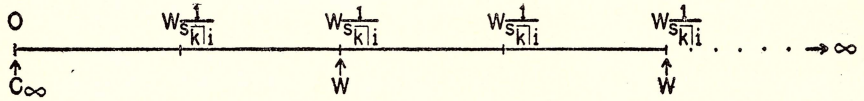


Fig. 34

The formula for capitalized cost then may be derived merely by showing the sum of the original cost of the asset and a fund to provide sufficient interest periodically as a deposit in a sinking fund to accumulate W every k periods. The fund necessary to provide perpetually the periodic sinking-fund deposit, $W \cdot \frac{1}{s_{\overline{k}|i}}$, is merely the present value of a perpetuity, or $\frac{W \cdot \frac{1}{s_{\overline{k}|i}}}{i}$. It should be noted that the rent, $W \cdot \frac{1}{s_{\overline{k}|i}}$, is needed in perpetuity because at the end of each k periods the sinking fund accumulated to that point in time is spent.

Capitalized-Cost Formulas:

$$C_\infty = O + \frac{W \cdot \frac{1}{s_{\overline{k}|i}}}{i} = O + \frac{W}{i} \cdot \frac{1}{s_{\overline{k}|i}}.$$

If O and W are of equal value, then the equation may be written:

$$C_\infty = W + \frac{W}{i} \cdot \frac{1}{s_{\overline{k}|i}} = W\left(1 + \frac{1}{i} \cdot \frac{1}{s_{\overline{k}|i}}\right) = \frac{W}{i}\left(i + \frac{1}{s_{\overline{k}|i}}\right) = \frac{W}{i} \cdot \frac{1}{a_{\overline{k}|i}}.$$

Here again it may be noted that the first of these equations is composed of five variables and the second of four. Only one direct application of capitalized cost will be illustrated.

Capitalized cost—What is the amount of a fund sufficient to finance the construction and equipping of a $500,000 hospital and provide for the renewal of $100,000 worth of equipment every 10 years? Assume money is worth 4% per annum.

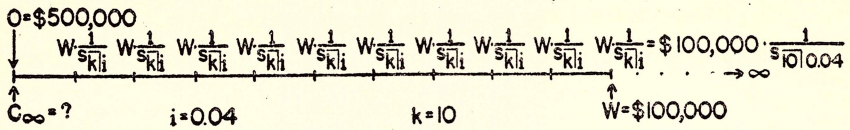


Fig. 35

a $C_\infty = O + \frac{W}{i} \cdot \frac{1}{s_{\overline{k}|i}}$

b $C_\infty = \$500{,}000 + \frac{\$100{,}000}{0.04} \cdot \frac{1}{s_{\overline{10}|\,0.04}}$

c $C_\infty = \$500{,}000 + (\$2{,}500{,}000) \cdot (0.08329)$

d $C_\infty = \$708{,}225.00.$

The concept of capitalized cost also may be utilized effectively to determine which of two or more assets are more economical when they perform the same function equally well but have different costs and different probable lives.

Comparative capitalized costs—A taxicab company has been using cabs that cost $900 each and these must be replaced every 4 years for $700. How much could the company pay for cabs that would last 6 years with a salvage value of $250 if money is assumed to be worth 7% converted semi-annually?

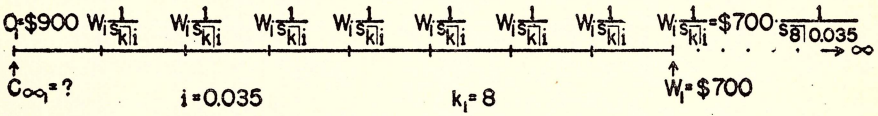


Fig. 36

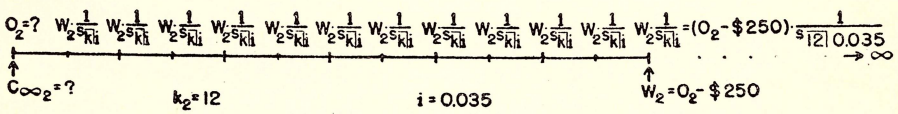


Fig. 37

a $C_\infty = O + \frac{W}{i} \cdot \frac{1}{s_{\overline{k}|i}}$ (for Figs. 36 and 37)

b $C_{\infty_1} = \$900 + \frac{\$700}{0.035} \cdot \frac{1}{s_{\overline{8}|\,0.035}}$ (Substituting in a from 36)

c $C_{\infty_2} = O_2 + \frac{O_2 - \$250}{0.035} \cdot \frac{1}{s_{\overline{12}|\,0.035}}$ (Substituting in a from 37)

d Let b equal c; then

$$O_2 + \frac{O_2 - \$250}{0.035} \cdot \frac{1}{s_{\overline{12}|\,0.035}} = \$900 + \frac{\$700}{0.035} \cdot \frac{1}{s_{\overline{8}|\,0.035}}$$

$$0.035\,O_2 + (O_2 - \$250)(0.06848) = (0.035)(\$900) + (\$700)(0.11048)$$

$$0.035\,O_2 + 0.06848\,O_2 - \$250(0.06848) = \$31.50 + \$77.34$$

$$0.10348\,O_2 - \$17.12 = \$31.50 + \$77.34$$

$$0.10348\,O_2 = \$31.50 + \$77.34 + \$17.12 = \$125.96$$

$$O_2 = \$1217.24$$

TEST YOUR KNOWLEDGE OF CAPITALIZED COST

46 A tennis court costing \$5,000 was given to a college. In addition to the original cost, it was estimated that \$3,000 would be required every 15 years to recondition the court. How much should the gift be, if, in addition to these, a fund is established to provide \$900 a year for maintainance? Assume interest earned on the funds at 3% per annum.

47 A British consol pays \$2,000 per year in perpetuity. What is the value of the consol if the market rate of interest is (a) 6%, (b) 5%, (c) 4%, (d) 2%?

48 What is the value of a farm which has a net annual income of \$10,000 if interest is 6%?

49 A city engineer is attempting to decide between two types of paving surface which are equally satisfactory for the purpose at hand. A costs \$13.50 per square yard with an estimated life of 25 years; B costs \$9 per square yard with an estimated life of 15 years. Which should he choose? Assume money worth 5%.

Amortization schedules

The manager of an installment finance business wishes a schedule prepared for a 12-months' contract to liquidate a debt for \$300 at an interest rate of $1\frac{1}{2}\%$ per month.

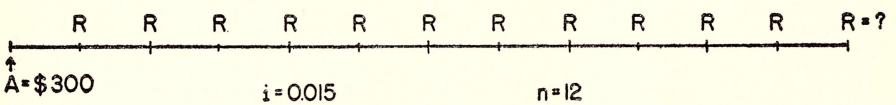


Fig. 38

a $$R = A \cdot \frac{1}{a_{\overline{n}|i}}$$

b $$R = \$300 \cdot \frac{1}{a_{\overline{12}|0.015}}$$

c $$R = \$300\ (0.09168) = \$27.50.$$

AMORTIZATION SCHEDULE

Period	Amount of Loan at Beginning of Period	Interest Earned During Period	Periodic Rent Necessary to Liquidate Debt	Amount Available to Reduce Debt
1	\$300.00	\$4.50	\$27.50	\$23.00
2	277.00	4.16	27.50	23.34
3	253.66	3.81	27.50	23.69
4	229.97	3.45	27.50	24.05
5	205.92	3.09	27.50	24.41
6	181.51	2.72	27.50	24.78
7	156.73	2.35	27.50	25.15
8	131.58	1.97	27.50	25.53
9	106.05	1.59	27.50	25.91
10	80.14	1.20	27.50	26.30
11	53.84	0.81	27.50	26.69
12	27.15	0.41	27.50	27.09

Note that the amount available to reduce the debt (\$27.09) is not equal to the amount of the loan at the beginning of the period (\$27.15). However, the difference is only 6 cents, which is just one-half the number of periods. Usually the difference in cents should be considerably less than one-half the number of periods but in no case should the difference in cents exceed one-half the number of periods.

COMPARISON BETWEEN THE SINKING-FUND AND AMORTIZATION METHODS OF CAPITAL FINANCING

Many capital loans both by industry and by political subdivisions are made some upon an amortization method and others upon the basis of a straight loan with a sinking-fund accumulation to ensure payment of the loan when due.

Illustrative Example

A corporation treasurer is interested in knowing the difference in the semi-annual cost by each of these two methods. Assume that the corporation wishes to borrow \$1,000,000 to be repaid in 10 years. Assume further that the market rate of interest for the type of risk which it represents is 5% converted semi-annually. Still further assume that a trust company has agreed to accept the semi-annual deposits under the sinking-fund arrangement at a rate of 3% per annum converted semi-annually.

Amortization Plan

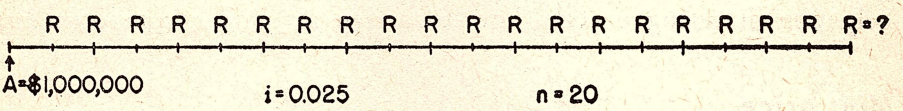


Fig. 39

a $$R = A \cdot \frac{1}{a_{\overline{n}|i}}$$

b $$R = \$1{,}000{,}000\ (0.06415) = \$64{,}150,$$

the semi-annual payment necessary to liquidate loan principal and interest in 10 years.

Sinking-Fund Plan

It is apparent that the regular semi-annual interest payment due to those from whom the corporation borrowed would be \$25,000. To this should be added the semi-annual payment necessary to accumulate \$1,000,000 into the sinking-fund. This payment may be derived as follows:

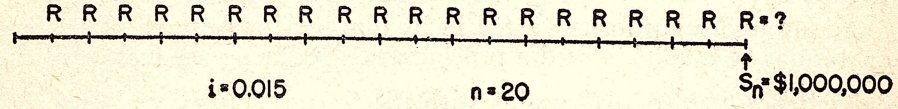


Fig. 40

a $$R = S_n \cdot \frac{1}{s_{\overline{n}|i}}$$

b $$R = \$1{,}000{,}000\ (0.04325) = \$43{,}250$$

Thus, the sinking-fund plan would cost \$68,250 semi-annually compared to \$64,150 by the amortization plan.

MATHEMATICS APPLIED TO MONEY AND BANKING

By Kenneth Lewis Trefftzs, Ph.D.

FOR our final article on the mathematics of finance, we come to a discussion of mathematics as applied to money and banking. The average reader of the financial columns in the daily newspapers and of economics articles in current magazines has been made familiar with most of the expressions which are introduced, but probably few have more than a hazy idea as to exactly how the processes to which the terms refer are carried out.

BANKING FORMULAS AND DEFINITIONS

Before undertaking to deal with problems of converting monies from one standard of currency to another or of computing the value of stocks and bonds, one must have some understanding of the principles employed by bankers, arbitragers, and brokers. A knowledge of the line of reasoning which underlies certain transactions will help one in seeing exactly how to arrive at similar computations.

Mint price

In the discussion of monetary standards, the term, *mint price*, is commonly used to indicate the price for the monetary metal (gold or silver) at the country's treasury. From 1837 to 1933, the dollar of the United States was composed of 0.900 fine gold and 0.100 alloy, and the total weight of the coin was 25.8 grains. The dollar, therefore, contained 23.22 grains of pure gold (90% of 25.8). A troy ounce of gold or silver weighs 480 grains troy. Thus, an ounce of gold furnished sufficient metal to coin 20.67 dollars.

$$\frac{480}{23.22}=\$20.67.$$

In this case, $20.67 is referred to as the mint price of gold.

Acting under authority vested in by him by Congress, the President on January 31, 1934, issued a proclamation reducing the weight of the dollar from 25.8 to $15\frac{5}{21}$ grains with the fineness unchanged at 0.900. Thus, the pure gold content of the dollar was reduced from 23.22 grains to 13.71 (actually 13.71428) grains. The new mint price is:

$$\frac{480}{13.71428}=\$35.00.$$

Since the pure gold content of the dollar was decreased from 23.22 to

13.71, it is sometimes said that we devalued the dollar by a total of 40.95 per cent.

$$(23.22-13.71)\div 23.22=40.95 \text{ per cent.}$$

The same fact is sometimes stated in another way by saying that the price of gold was raised by 69 per cent.

$$(\$35-\$20.67)\div \$20.67=69 \text{ per cent.}$$

Both are true statements, but one is based on price, the other on weight.

Mint and market ratio

In 1792, Congress adopted a bimetallic system; rather than one standard money, we had two, both gold and silver. Under this Act, the gold dollar contained 24.75 grains of pure gold. (What was the mint price?)

By the same Act, the silver dollar was to contain 371.25 grains of pure silver. The mint price of silver was thus fixed at

$$\frac{480}{371.25}=\$1.29.$$

It can be seen that the ratio between the two metals at the mint was 15 to 1. This ratio is called the mint ratio; it is found by comparing either the weight of the two standard monies $\left(\frac{371.25}{24.75}\right)$ or the mint prices $\left(\frac{\$19.39}{\$1.29}\right)$ of the two.

When in 1837 the weight of the gold in the dollar was fixed at 23.22 grains, and the pure silver content of the silver dollar was unaltered, a mint ratio of $\frac{371.25}{23.22}$ (equal to 15.99 to 1) was established. The mint price is fixed by law and continues until changed by the necessary action of the legislature.

Because gold was generally used as a monetary metal during the period from 1834 to 1934, the market price throughout the world continued until 1934 at about $20.67 or its equivalent, but during the same period the market price of silver in terms of gold fluctuated with changes in the supply and demand. When the market price of silver fell below $1.29 an ounce, holders of silver bullion preferred to sell it to the mint at $1.29 an ounce.

The relation between the price of silver and the price of gold in the market is referred to as the market ratio. Whenever the price of silver dropped below $1.29 an ounce, this ratio would rise above the mint ratio, and silver would flow to the mint. Thus, if the price of silver were $1.20 an ounce in the market and gold stood at $20.67, the market ratio would be

$$\frac{\$20.67}{\$1.20}=17.22 \text{ to } 1.$$

If the price of silver rose to $1.35 an ounce, silver would flow to the market, and silver dollars could be profitably sold as bullion.

One silver dollar would bring \$1.04 in the bullion market, since

$$\frac{371.25}{480}\times\$1.35=\$1.04.$$

Under such circumstances, the market ratio would be 15.3 to 1, since

$$\frac{\$20.67}{\$1.35}=15.3,$$

which is under the mint ratio. When such a condition prevails, the silver is said to be *undervalued* for circulation purposes and tends to disappear from circulation.

Recently, there has been much demand on the part of certain senators for the re-adoption of the bimetallic system at a ratio of 16 to 1.

Mint par of exchange

When all nations have a monetary unit defined in terms of a fixed quantity of gold, then each currency bears a constant relationship to every other currency, and for all intents and purposes an international currency comes into being. For this reason, the gold standard is often referred to as the international gold standard.

Before 1934, the United States dollar weighed 25.8 grains and was only 0.900 fine. At the time Great Britain went off the gold standard in 1931, the weight of the British monetary unit, the pound sterling, was fixed at 123.274 grains of standard gold. Standard gold is $\frac{11}{12}$ fine. Although the two monetary units differed in fineness, it is easy to find the pure gold content:

$$\frac{25.8\times9}{10}=23.22 \text{ and } \frac{123.274\times11}{12}=113.00116;$$

and it was the pure gold content which determined their relative values. When the pure gold contents are compared, we speak of the quotient as being the *mint par of exchange*. In this case, it would be

$$\frac{113.00116}{23.22}=4.8665.$$

In pure gold content, the pound sterling was equivalent to approximately \$4.87. This figure, \$4.87, is usually referred to simply as the *par of exchange*, or *parity*. In the same way, when both countries compared were on a gold standard, the relation between dollars and francs, pounds and marks, marks and francs, and dollars and marks was relatively fixed. Between any two currencies based on and readily convertible in gold, a fixed ratio existed. This mint par of exchange depended, in the final analysis, not on the coins themselves but on the legal definitions of the coins; in many instances, the coin itself might not even be minted. Nonetheless, a ratio could be determined which remained the same until the legal definition of one of the coins was changed.

TEST YOUR KNOWLEDGE OF MINT AND MARKET PRICES

1 If our gold dollar contains 13.71 grains of fine gold, and if we should adopt the bimetallic standard at a ratio of 16 to 1, what would be the pure silver content of the new silver dollar?

2 What would be the mint price of silver?

3 The world market price of silver is about 45 cents a troy ounce under war-time conditons; the price of gold, $35 an ounce. What is the present market ratio?

4 Since the gold content of the dollar has been reduced to 13.71 grains, what is the mint par of exchange with the pound sterling when the pound sterling contains 113.00116 grains of pure gold? What would be the gold content of the Portuguese escude if the mint par of exchange between it and the dollar was 7.4831 cents?

CHAIN RULE

In exchange computations, use is often made of what is referred to as the *chain rule*, a simple arithmetical operation of connecting a series of simple equations and establishing a comparison between two quantities not directly related. In order to find the number of dollars equal to a pound on the basis of the data given in the preceding problem, we should proceed as follows, using the chain rule.

The first link of the chain must be the unknown quantity, and the last link must be in the same denomination. In the example given, this is dollars. The left-hand side of each equation must be in the same denomination as the right-hand side of the preceding equation. If the first link of the chain is the unknown quantity, it will then be equal to the product of all the numerical links on the right, divided by the product of all the numerical links on the left.

Illustrative Example A

How many dollars equal one pound?

$$£1 = 123.274 \text{ grains standard.}$$

$$12 \text{ grains standard} = 11 \text{ grains fine.}$$

$$9 \text{ grains fine} = 10 \text{ grains } \frac{9}{10} \text{ fine}$$

$$25.8 \text{ grains } \frac{9}{10} \text{ fine} = \$1.$$

$$\frac{123.274 \times 11 \times 10 \times 1}{12 \times 9 \times 25.8} = \$4.8665.$$

Illustrative Example B

How many francs equal one pound?

Prior to the devaluation of the franc in 1936, and the devaluation of the dollar in 1934, the dollar contained 25.8 grains (or 1.67177 grams)

of gold $\frac{9}{10}$ fine; the franc, $65\frac{1}{2}$ milligrams of gold $\frac{9}{10}$ fine; the British pound, 7.98805 grams of standard gold.

£1 = 7.988 grams standard.

12 grams standard = 11 grams fine.

9 grams fine = 10 grams $\frac{9}{10}$ fine.

65.5 grams $\frac{9}{10}$ fine = 1,000 francs.

$$\frac{7.988 \times 11 \times 10 \times 1{,}000}{12 \times 9 \times 65.5} = 124.21.$$

Therefore, £1 equals 124.21 francs.

Illustrative Example C

How may dollars equal one franc?

Both basic coins are defined as given weights of gold, $\frac{9}{10}$ fine; therefore, it is not necessary to find their pure gold content since the relationship between the total weight is the same as the relationship between their pure gold contents. This relationship may be expressed as follows:

$$\frac{0.0655 \text{ grams in 1 franc}}{1.67177 \text{ grams in \$1}} = 0.03918.$$

Thus, it is seen that one franc was equal to 3.918 cents. The accuracy of this calculation may be checked in the following manner:

£1 = 124.21 francs

1 franc = 3.918 cents.

The number of francs in a pound sterling, times the number of cents in a franc, should equal the number of cents in a pound sterling.

$$124.21 \times 0.03918 = \$4.8665.$$

Thus, we find that the pound is equal to \$4.8665. Since we found the pound equivalent to the same amount in calculating the ratio between the dollar and the pound, we can assume that the calculated ratio between the dollar and the franc is correct. If one franc is equal to 3.918 cents, then \$1 must be equal to $\frac{\$1}{0.03918} = 25.52$ francs.

Gold points

The fact that currencies bear a fixed ratio to one another does not mean that they will circulate freely in either country. The significance of the gold standard as an international standard is not that the coins of one nation will pass freely in another country, but that a fixed relationship exists between their monies. This fixed relationship has its greatest significance in foreign exchange; that is, in exchanging one currency into another or in making remittances abroad. Thus, an American importer owing £205,486.49 and wishing to make payment in England could ship gold, and the English creditor could accept it by weight and have it converted into English money . When both countries

were on a gold standard, the cost of shipping gold would have been somewhat as follows:

Cost of gold coins (£205,486.49 at $4.8665)	$1,000,000
Ocean freight	2,500
Bullion brokerage	250
Insurance	450
Packing	50
Handling	290
Treasury commission	2,500
Bill of lading	50
Total cost	$1,006,090

The actual cost to him to make such a payment would have been $1,006,090, or the equivalent of $4.896 for each pound sterling he paid.

$$\frac{\$1,006,090}{£205,486.49}=4.896.$$

This figure, 4.896, is the *gold export point*, and rather than pay more than this amount, the importer would ship gold to satisfy his obligation.

The lower limit on exchange fluctuations is determined in a similar manner. Assuming shipment equivalent to $1,000,000, the *gold import points*, or lower limit of exchange fluctuations, if the conditions listed below prevailed, would be calculated as follows:

Cost of the equivalent of $1,000,000 in pounds sterling at $4.8665	£205,486.49
Bullion brokerage	50
Ocean freight	515
Insurance	90
Packing	10
Handling	65
Treasury commission	515
Melting charge	5
Customs entry	1
	£206,737.49

Under the conditions given, in order to transport the equivalent of £205,486.49 in gold, we must incur a cost of £1,251. The net value of the pounds received is equivalent to $4.837 a pound. A total expenditure of £206,737.49 was made to ship the equivalent of $1,000,000.

$$\frac{\$1,000,000}{£206,737.49}=4.837.$$

This figure, 4.837, would be called the *gold import point*, and would establish the lower limit for fluctuations in the exchange rates between the two countries as long as the conditions listed above prevailed.

It should be noted, however, that any increase in the expense of shipping gold increases the spread—that is, raises the gold export point and lowers the gold import point—while any decrease narrows the difference between import and export points.

CONVERSION OF FOREIGN MONIES

Foreign-exchange rates may be defined as prices at which foreign exchange is bought and sold. To buy "foreign exchange" on London, for instance, is simply to buy a draft, or bill of exchange, ordering someone, usually a bank, to pay a certain sum in pounds sterling in London to the designated payee. The first calculation

which might arise in this connection is the conversion of one money into another. This is a relatively simple matter when dealing with your own currency, but at first glance it appears more difficult when dealing with two foreign currencies. A few examples, however, will show the relative simplicity of such conversions.

In dealing with English money, it is necessary to know that:

12 pence (12*d.*) = 1 shilling (1*s.*) and 20 shillings (20*s.*) = 1 pound (£1).*

Illustrative Example A

Convert $22.80 into sterling at the rate of £1 = 4.50.

STEP	EXPLANATION	COMPUTATION
a	Divide 22.80 by 4.5; multiply quotient, 5, by 4.50; subtract.	£5 1*s.*4*d.* 4.50) 22.80 22.50
b	The remainder of 30 obviously cannot be converted into a whole pound. The next step is to multiply the remainder, 30, by the number of shillings in a pound.	30 ×20 600
c	Continue the division, with the quotient, 1, stated in shillings.	450
d	Division cannot continue in units of shillings.	150
e	Multiply the remainder, 150, by the number of pence in a shilling.	×12 1800
f	Continue the division, with the quotient, 4, stated in pence.	1800

Thus, it is seen that the amount will convert at the given rate into £5 1*s.*4*d.*

The process of converting sterling into dollars is not fundamentally different, as the following example shows:

Illustrative Example B

Convert £50 18*s.*4*d.* into dollars, when the pound is quoted at $4.80.

STEP	EXPLANATION	COMPUTATION
a	Convert pounds to dollars.	£50 × $4.80 = $240.00
b	Convert shillings to fractional parts of a pound and then into dollars.	$18s. \times \frac{\$4.80}{20} =$ 4.32
c	Convert pence into a fractional part of a shilling, then into fractional parts of a pound, and then into dollars.	$4d. \times \frac{\$4.80}{20 \times 12} =$ 0.08
d	Add.	$244.40

* These are the basic monies of England, and all that one needs to know for purposes of conversions. There are other common monetary measures. Some are coined: the crown, which is the equivalent of 5 shillings; the sixpence, the equivalent of $\frac{1}{2}$ shilling; the halfpenny, the equivalent of $\frac{1}{2}$ penny; and the farthing, the equivalent of $\frac{1}{4}$ penny. The guinea, which is at present purely a monetary measure, is the equivalent of 21 shillings and is still used occasionally for certain fees. *Pence* is the plural of the word, *penny*.

Practical suggestions

When it is necessary to make constant conversions, the experienced calculator may save considerable time by reducing shillings and pence to the decimal parts of a pound and making up tables to which he can quickly refer. (See Table XCV.) Under this method, to convert the foreign value into the domestic value, if dealing with British money, reduce shillings and pence to decimals of a pound and multiply by the dollar value per pound.

Illustrative Example A

To convert £1,200 8*s*.6*d*. to dollars when the pound is quoted at $4.80:

Pounds	1,200.000
$8s. = \frac{8}{20}$ of a pound, or $\frac{4}{10}$	0.400
$6d. = \frac{6}{12}$ of $\frac{1}{20}$ of a pound, or $\frac{1}{40}$	0.025
Stated in pounds, the entire amount is	1,200.425 pounds
The conversion rate per pound is	4.80
The product is	$5,762.04.

In converting a domestic value to pounds, divide the amount by the dollar value of the pound and reduce decimal parts of a pound to shillings and pence.

Illustrative Example B

Convert $22.80 to pounds at the rate of £1 = $4.50.

Step	Explanation	Computation
a	Convert to pounds and fraction of a pound.	$\frac{\$22.80}{4.50} = £5.0667$
b	The fractional part of a pound, 0.0667, can be converted into shillings by multiplying it by 20.	0.0667 ×20 = 1.334 shillings
c	The fractional part of a shilling, 0.334, is converted to pence by multiplying it by 12; the minor fraction of $\frac{8}{1000}$ is disregarded.	0.334 ×12 = 4*d*.
d	Hence, the total is: £5 1*s*.4*d*.	

When large numbers of conversions are to be made, the logical procedure is to compile a table of the decimal equivalents of a pound. Such a table has been computed and is given on page 1179. By reference to it, one readily sees that 8*s*.6*d*. is equivalent to 0.425 of a pound, just as we have calculated above. In keeping accounts of English transactions, however, the practice continues of carrying three separate columns for pounds, shillings, and pence.

Preparing a conversion table

Whenever it is necessary to calculate more than a relatively small number of conversions of a domestic into a foreign currency, or a foreign into a domestic currency, time can be saved by preparing an exchange table. The preparation of such a table between the Mexican peso at 19.3375 cents and the United States dollar would involve the following steps:

a Make a table with the headings as given in Table a.

b Fill in columns 1 and 3, from 1 to 9.

c In column 2, we are given the number, 19.3375, which is the number of United States cents equal, at the current rate, to a Mexican peso. Two pesos equal twice this amount; three pesos, three times; etc. Thus, it is not difficult to multiply 19.3375 cents by each digit up to 9, and complete column 2.

d Column 4 can be filled in by dividing $1 by 19.3375. The quotient is the number of pesos equivalent to one United States dollar at the rate of 19.3375 cents. The balance of the column is found by multiplying this quotient by each digit up to 9 and filling in the appropriate space in the column.

TABLE a

EXCHANGE TABLE

Pesos to Dollars—Dollars to Pesos
Mexican Peso=19.3375 cents

1 Pesos	=	2 Dollars	3 Dollars	=	4 Pesos
1		0.193375	1		5.17130
2		0.386750	2		10.3426
3		0.580125	3		15.5139
4		0.773500	4		20.6852
5		0.966875	5		25.8565
6		1.160250	6		31.0278
7		1.353625	7		36.1991
8		1.547000	8		41.3704
9		1.740375	9		46.5417

With the use of the table, rapid conversions can be made.

Illustrative Example

Determine the price of 974 pesos when 1 peso equals 19.3375 cents:

900 pesos = $174.0375 (Row 9×100)
70 pesos = 13.53625 (Row 7× 10)
4 pesos = 0.77350 (Row 4)

$188.34725, or $188.35.

Usually, when exchange rates fluctuate from day to day within relatively narrow margins, reference need be made to only a small number of tables; but when violent fluctuations are common, the number of tables necessary becomes unwieldy.

INTEREST ON FOREIGN MONEY

When it is necessary to calculate interest on foreign money, one method is to follow the procedure outlined in a preceding section (p. 1129):

Illustrative Example

What is the interest on £325 5*s*. 6*d*. at 3 per cent for sixty days?

Pounds	325
Shillings $\left(5s.=\frac{1}{4}\text{ pound}\right)$	0.250
Pence $\left(6d.=\frac{1}{2}\text{ of }\frac{1}{20}\text{ of a pound}\right)$	0.025
	£325.275*

a Interest for 60 days at 6 per cent. Point off two places. £3.25275

b Interest for 60 days at 3 per cent $\left(\frac{1}{2}\text{ of 6 per cent}\right)$ £1.626375

c 0.626375 of a pound, multiplied by 20 = 12.5275 shillings
0.5275 of a shilling equals (multiply by 12) 6.33 pence
Therefore, the interest is equal to £1 12*s*. 6*d*.

Second Method

What is the interest on £325 5*s*. 6*d*. at 3 per cent for 60 days?

Step	Explanation	Computation
a	Number of pounds	325
	Multiply by number of shillings per pound	×20
	Pounds reduced to shillings	6500
	Add number of shillings in principal	+ 5
		6505 shillings
	Change to pence by multiplying by number of pence in shilling	×12
	Shillings reduced to pence	78060
	Add number of pence in principal	+ 6
	Total number of pence in principal	78066 pence
b	Interest for 60 days at 6 per cent	780.66 pence
	Interest for 60 days at 3 per cent (one-half the above)	390.33 pence

c 390.33 could be considered as $\frac{390}{12}$, or 32 shillings, 6 pence.

$\frac{32}{20}$ equals 1, with a remainder of 12.

Hence, the total interest equals £1 12*s*. 6*d*.

* This answer may be verified by referring to Table XCV on page 1179.

TEST YOUR KNOWLEDGE OF THE CONVERSION OF MONIES

5 In 1939, the lira was equivalent to 5.0568 cents, and the British pound worth 4.6525 dollars. By use of the chain rule, find the value of the pound in lire.

6 If the rate of exchange in New York on London is $\$4.46\frac{3}{8}$, what is the cost of a demand draft for £150 7*s*.6*d*.?

7 If the rate of exchange is $\$4.46\frac{3}{8}$, what is the maximum draft which can be purchased with $1,000?

Using Table XCV, convert the following into decimal parts of a pound:

	£	*s*.	*d*.		£	*s*.	*d*.		£	*s*.	*d*.
8	1	12	11	**10**	–	8	8	**12**	9	10	9
9	5	5	7	**11**	1	–	11	**13**	5	5	–

Using the same table, convert the following equivalents into pounds, shillings, and pence:

14 5.24167 **16** 9.99583 **18** 1.00833
15 211.5 **17** 23.62500 **19** 0.300000

20 Convert $493 into pesos at the rate of one peso equals 19.3375 cents.
21 Make an exchange table for dollars and Argentine pesos (at rate of 23.7525 cents) for any sum up to 1,000,000 pesos or dollars.
22 From the table prepared for problem 21, determine the price of 369 pesos.
23 From the table prepared for problem 21, convert $852 into pesos.
24 Find the interest on £200 5*s*.6*d*. for 63 days at 6 per cent.
25 The yield on 90-day bankers' bills is 0.63 per cent per annum. Find the interest on £700 19*s*.11*d*. invested in such bills, for 90 days.

Cable rates

The typical pre-war quotations found in most of the larger newspapers of New York and other leading cities formerly quoted exchange on London in New York somewhat as follows:

Sterling	
Demand	$4.68\frac{3}{16}$
Cable	$4.68\frac{1}{4}$
Com. 60 day	$4.66\frac{15}{16}$
Com. 90 day	$4.66\frac{9}{16}$.

Usually sterling cables are generally slightly higher than demand sterling. As a general rule, it can be said that cables in any foreign exchange are slightly higher than demand exchange in the same currency. Cable exchange differs from demand exchange in that it reaches the payee much sooner after purchase. The time it takes for such a payment to reach England is usually one day or less; and other countries, two days after purchase. A demand draft sold by an American bank for payment abroad will not ordinarily be deducted from the banker's balance in England for a week, so that in effect the selling bank has the use of its funds here as well as abroad for seven days. When

interest rates are low and mail rapid, cable rates and demand rates tend toward equality. The buyer of cable transfer bears the costs of the cable.

The difference between the cable rate and the demand rate merely *tends* to vary by the interest charges; other factors of equal importance are involved. Market conditions are determined, to a certain extent, by the balances held abroad by the United States banks. If the balances are large, or if they are nearing the maximum amount which the United States bankers desire to maintain abroad, the purchase of sight drafts or cables may be encouraged by lowering the respective rates. Telegraphic transfers and sight drafts were commonly used during the 1930's.

Time bills

Time bills are also commonly used to pay foreign obligations. If they have not more than 30 days to run, they are often referred to as *short bills*; the term, *long bill*, is commonly used if they have more than 30 days to run. The period of the bill is called the *usance* and varies with the custom of the trade. The longer the distance, usually the greater the usance. In much of the trade, long bills, running from 60 to 90 days, are used. In times of unsettled world conditions, the use of long bills declines.

The relationship between cable rates and time rates appears more complex than that between demand bills and cable rates. In the list of rates given on page 1132, 60-day drafts are quoted at $4.66\frac{15}{16}$, or $1\frac{5}{16}$ less than the cable rate; 90-day commercial drafts are $\frac{3}{8}$ of a cent a pound less than 60-day drafts, or $1\frac{11}{16}$ cents less than the cable rate.

Under ordinary conditions, it will be found that the longer the life of the draft the lower the rate, relative to the cable rate. This relationship exists between the cable rate and the time rate on any country. The rate of exchange on 60-day drafts is lower than the cable rate by the current rate of discount on 60-day funds; and the 90-day rate is lower by the current rate of discount on 90-day funds. The rate of exchange on time drafts is governed by the current rate of discount prevailing in the market in whose currency the draft is drawn. Thus, the rate of exchange on a 60-day sterling draft drawn in New York on London will be controlled by the London rate of discount on 60-day funds, since the bill will be discounted there.

Illustrative Example

Assume that the cable rate in New York on London is $4.85656 and that the discount rate in London on 90-day bills is 2 per cent. Since

3 days of grace are allowed in London, one must calculate the discount at 2 per cent, not for 90 days but for a total of 93 days. The calculation would be as follows:

Cable rate	\$4.85656
Discount at 2 per cent for 93 days	−0.02509
Tax $\left(\text{about } \frac{1}{20} \text{ of 1 per cent}\right)$	−0.00242
Buying rate per pound for 90-day bill (approximately)	\$4.82905

It is not difficult to see why this relationship between the cable rate and the rate on 90-day bills would be maintained during periods of relatively stable exchange rates. An American importer who has bought goods for which he must pay in 90 days has in effect the option of buying today a 90-day bill and remitting it to his creditors or waiting until the day of maturity and paying by cable transfer. If the American buys the 90-day draft, he is deprived of the use of his funds for 90 days, and the banker who sells him the draft has the use of them. The buyer, therefore, is not willing to pay more than the discounted value of the cable rate. If the rate differs by less than the interest, it pays a banker to sell bills and cover by cable on the due day.

Calculation of price of bill

In order to avoid the risks of fluctuating exchange, an American exporter quotes his price in dollars and draws his bills in dollars against a foreign importer or an importer's bank. The important problem faced by the American exporter in extablishing the price for his merchandise is to know at what rate the bank will discount his bill, and what additional charges he may have to meet. If an exporter wants to net \$10,000 on the cargo, and by the terms of the sale he is to bear the freight of \$1,500 and marine insurance of \$100 and is to allow the English importers 60 days to pay from date of sale, he must ask not \$11,600, since

$$\$10{,}000+\$1{,}500+\$100=\$11{,}600,$$

but also enough to cover the discount for 60 days on the face of the draft, plus the banker's collection charge (which we shall assume to be $\frac{1}{4}$ of 1 per cent of the face). If the bank has agreed in advance to discount the bill at 3 per cent, then the final calculation is as follows:

$$100 \text{ per cent} - \left[\left(\frac{60}{360}\times 0.03\right) + \frac{1}{4} \text{ per cent of the face of the draft}\right] = \$11{,}600$$

$$99.25 \text{ per cent} = \$11{,}600$$

$$100 \text{ per cent} = \$11{,}687.66, \text{ face of the draft}$$

$$\frac{60}{360}\times 0.03 \times \$11{,}687.66 = \$58.44, \text{ discount}$$

$$\frac{1}{4} \text{ per cent} = 0.0025 \times \$11{,}687.66 = \$29.22, \text{ collection charge.}$$

If the American exporter quotes his price in the foreign currency to avoid the risks of exchange fluctuations, he should procure a committal from his bank as to the rate at which it will buy the foreign bills when drawn and offered. Assuming the sight (that is, the demand) rate of exchange to be \$4.6825, the banker computes his offering price on the basis of what he can get for the draft in London when it arrives there. He calculates this price by deducting interest for 60 days, plus the 3 days of grace, at the rate in London, plus the British stamp tax, and the British bank's commission, as well as such profits as he himself anticipates.

Sight rate on London	\$4.6825
Discount for 63 days at $\frac{3}{4}$ per cent	−0.0061
Stamp tax	−0.0024
Commission to British banker	−0.0050
Profit to American banker	−0.0025
Offering rate of American bank for draft	\$4.6665

Since the exporter wants to net \$10,000 on the sale, the total draft must be the equivalent of \$10,000 plus freight of \$1,500 plus insurance of \$100, or a total of \$11,600. The amount of the draft drawn in English currency then would be $\frac{\$11{,}600}{4.6665} = £2{,}485\ 16s.\ 1d.$

Forward exchange

The discussion thus far has dealt with the market for trading in foreign exchange, known as the *spot market*, in which all exchange, whether cable, demand, or time, is for immediate delivery by the seller and immediate payment by the buyer. In addition to the spot market, foreign exchange may be bought and sold which requires future delivery by the seller and payment by the buyer. Foreign exchange *futures* are known as forward exchange, and the rates applying to them are referred to as forward-exchange rates.

The forward rates are quoted in the newspaper under the heading, "Ninety-day rates (Points)", and are quoted at a *discount*, at a *premium*, or at *parity*. If, on a given day, demand sterling is quoted at $4.68\frac{3}{16}$ and at the same time sterling for delivery 90 days hence is quoted at $1\frac{15}{16}$ discount, the forward rate on demand sterling is in effect $4.66\frac{1}{4}$. If sterling futures are quoted at a discount in New York, then dollar futures must be quoted at a premium in London.

Forward exchange is bought by two kinds of buyers. One type is represented by those who have a definite commitment to meet in the future and who do not care to assume the risks of fluctuating exchange

rates. An example of this type would be an American who plans to import woolens from England. If he can get them for £5,000, he can satisfactorily meet his contract. If the rate of exchange should rise before he pays for them 3 months hence, he will lose money. Therefore, it will pay him to buy future exchange at the rate given today, since he has then contracted for the necessary amount to meet his bill and need not worry about future fluctuations. He will pay for the futures at the time of delivery—*i.e.*, in 90 days.

The second type of buyer of forward exchange is the speculator. He deliberately goes into the market hoping to gain by changes in exchange rates. If he believes that the rate will rise, he buys; and if it does rise, he profits. For example, on May 2, 1936, the spot rate on London was \$4.94, the forward rate for 3 months was at a discount of $\frac{1}{2}$. A speculator anticipating a market rise in the future would buy pounds on this date for delivery August 1, at the rate of \$4.935. On August 1, the spot price of pounds was \$5.0175. The speculator would have made a profit of 8.25 cents a pound by selling his futures in August at the spot price.

Exchange arbitrage

In quoting exchange rates, the practice in the United States since World War I has been to quote the rates on foreign countries in terms of the amount of United States currency required to purchase one unit of the foreign currency. For example, the pound sterling might be quoted at $\$4.68\frac{3}{16}$; the French franc, at 2.23 cents; the Argentine peso, at 29.77 cents; the Swedish krona, at 23.80 cents. This, the *direct method* of quoting exchange, is widely used in such countries as the United States and France.

Another method used (for example, in Great Britain) is called the *indirect method*, since it states the number of units of foreign currency which can be had in exchange for a unit of the domestic currency. Thus, London quotes the dollar on the basis of how many dollars in exchange for a pound. (For example, the rate might be \$4.68 = £1. If the direct method of quotation were used, the rate would be cited as $\$1 = 4s.3\frac{1}{4}d.$)

The relation between direct and indirect quotations may be clarified somewhat by the following tabulation:

Between:	Quotation by New York (Direct)	Quotation by London (Indirect)	Quotation by Paris (Direct)
New York and London	4.6825	4.6825	....
New York and Paris	0.0220	....	45.45
London and Paris	....	212.84	212.84

A third method of quoting foreign exchange sometimes used is the *discount-and-premium method.* It is used primarily between currencies which have a mint par of unity. For example, the Canadian dollar is commonly quoted by banks at a premium above or a discount below the United States dollar. When the United States dollar is selling at $1.10 in Canadian money, it means that Canadian money is selling at what is generally called a discount of 10 per cent, whereas our money is selling at a premium. (Actually, if it were a discount of 10 per cent, $1 would buy $1.11.) The discount method is also used to some extent in England in quoting on countries, particularly members of the empire, which use a pound of the same legal definition as the pound sterling.

A close relationship must always exist between the rates quoted in two countries and among the rates quoted in three or more countries. For instance, if the cable rate in New York on London should be $4.68 while at the same time dollars should be quoted in London at $4.6775, the exchanges are not in balance. Observing such a discrepancy in the rates, an arbitrager—almost invariably a banker in very close touch with the market—would immediately undertake the following operation. In New York, he would sell £10,000 sterling cable at the quoted rate of $4.68, thus giving him possession of $46,800. At the same time, he would have his agent in London sell $46,775 cable for £10,000 in London. He would then use the proceeds of the American sale of $46,800 to cover the sale of dollars in London and the $46,775 proceeds of the sales of dollars in London to cover the sale of pounds sterling in New York, leaving him a gross profit of $25. He would use the £10,000 he received from the London sale to cover the cable on London sold in New York.

Under war conditions—and for that matter, since the early 1930's—, the treasuries or other public authorities of many countries have exercised strict control over the foreign-exchange market. Consequently, exchange rates have not had free play, and it is difficult at this time to predict what kind of post-war currency stabilization plans may be adopted. It may be that arbitrage between currencies will be entirely in the hands of public authorities. Since many monetary changes may occur in the future (even the names of the monetary units of two countries were changed in 1943), there is a possibility that we may have a new international monetary unit, such as the *unitas* suggested by the United States Treasury or the *bancor* suggested by the English monetary authorities. For example of arbitrage, then, we must look to the period preceding World War II. Under the freedom of exchange which then existed, bankers dealing in international exchange kept virtually constant quotations during banking hours on the buying and selling rates in all important currencies. Under such circumstances, a close relationship always existed among the currencies of the world. This is illustrated as follows:

Assume, as a simple example, that London quotes dollars at \$4.80 to the pound and francs at 96 to the pound. The rate for pounds quoted by New York and Paris should be extremely close to 4.80 and 96, respectively. At the same time, the rate quoted by New York for francs will be close to 5 (since $\frac{4.80}{96}=5$) and that quoted by Paris for dollars close to 20 (since $\frac{96}{4.80}=20$).

Illustrative Problem

Suppose you are in charge of the foreign-exchange department of a large bank, constantly buying and selling exchange; and the market conditions, which have been as described in the preceding paragraph, change to the following figures:

In New York		In London	
Rate on Paris	\$0.05	Rate on New York	\$4.80
Rate on London	4.80	Rate on Paris	0.94

How could you profit from an arbitrage?

a In New York, buy cables on Paris for Fr. 47,000 at a cost of \$2,350;

b Direct your agent in London to sell Fr. 47,000 at 94 francs to the pound

$$\frac{47{,}000}{94}=£500;$$

c Sell cable in New York on London for £500 at \$4.80, with proceeds of \$2,400.

The cable purchased in New York on Paris would cover the francs sold by the London agent, and proceeds of the London sale on Paris would cover the draft on London sold in New York. Since the sale of the pounds would bring \$2,400, and the cost of the francs would be only \$2,350, the gross profit would be \$50. (The net profit would be somewhat less than this, and the total profit shown in the illustration has been exaggerated to simplify matters.) Usually the profit is small ($\frac{1}{5}$ of 1 per cent or less); but the arbitrager, taking the minimum of risk and operating in all markets simultaneously, is content with a very small profit.

The pressure exerted by such transactions in London tends to raise the franc (remember that the indirect method of quotation is used in London; when one says that the franc is raised, it means that the quotation of the pound in francs is increased—*i.e.*, the pound is weak).

Suppose that the cable quotations in New York are as follows:

New York on London	\$4.98
London on New York	4.98
New York on Amsterdam	0.676 per florin.

The quotation in Amsterdam is:

Florin 7.00 per pound sterling.

With \$10,000 in New York, how can one profit from arbitrage?

a Buy cable on Amsterdam for 14,792.89 florins;

b With 14,792.89 florins in Amsterdam, buy cable on London for £2,113.27;

c With sterling in London, buy cable on New York for \$10,524.08.

This would leave a profit of \$524.08 before expenses were deducted. Shortly before the beginning of World War II, the sterling quotations

in New York fluctuated around 4.6625 and the quotation on the franc was around 2.66 cents. With the exchange rates at these levels, one seeking to gain from arbitrage transactions would make up a table showing parities in dollars, francs, and pounds sterling. Using the chain rule explained previously (p. 1125), one can develop such a table by calculating the quotations on francs in London as follows:

Illustrative Example

How many francs equal £1?

£1 = $4.6625

$1 = 37.594 francs

$$£1 = \frac{37.594 \times 4.6625}{1 \times 1} = 175\frac{1}{4} \text{ francs } (0.28 \text{ rounded to } \frac{1}{4}).$$

In a similar manner, calculations can be made for sterling at 4.66, $4.66\frac{1}{4}$, $4.66\frac{1}{2}$, and $4.66\frac{3}{4}$, and for fluctuations in the London quotations of the franc from 175 to $175\frac{3}{4}$. With these figures, the following table can be constructed, showing that, when the New York quotation for sterling is $4.66 and the London quotation for francs is 175, the New York parity for francs is 2.6628 cents. If the market quotation differs from this figure, there is an opportunity for profitable arbitrage. In actual arbitrage work, a table with more minute gradations would be likely, since speed of calculation is essential.

TABLE b

PARITIES IN DOLLARS, FRANCS, AND POUNDS STERLING

£	175	175¼	175½	175¾
	$1	$1	$1	$1
4.66	2.6628	2.65906	2.65527	2.65149
4.66¼	2.6643	2.66049	2.65669	2.6529
4.66½	2.6657	2.66191	2.65812	2.65434
4.66¾	2.6671	2.66333	2.65954	2.65576

The mathematical relationship between three currencies can be shown as follows:

Let:

$m_{1,2}$ = the amount that center 1 is annually indebted to center 2;

$m_{2,1}$ = the amount that center 2 is annually indebted to center 1;

$C_{1,2}$ = par of exchange of center 1 on center 2;

$C_{2,1}$ = par of exchange of center 2 on center 1.

If the two centers balance their exchanges without transporting money, then

$$m_{1,2} \cdot C_{1,2} = m_{2,1}$$

or

$$C_{1,2} = \frac{m_{2,1}}{m_{1,2}}.$$

In general, then, $C_{2,1}$ is the reciprocal of $C_{1,2}$:

$$C_{1,2} = \frac{1}{C_{2,1}} \quad \text{or} \quad C_{2,1} = \frac{1}{C_{1,2}}$$

Let us assume that there are more than two centers and let m_{ik} express the total of the sums for which center i is indebted to center k over some fixed interval of time. Let C_{ik} be the rate of exchange between i and k. If, then, we consider the case of r centers, there will be $r(r-1)$ such relationships; but, because of the reciprocal relationship noted before, the number of independent ratios will reduce to $\frac{r(r-1)}{2}$. The coefficients are not all mutually independent, for, if it is found that

$$C_{ik} > C_{il} \cdot C_{lk},$$

then a person who needs to convey money from center k to center i will, instead of getting a draft from k on i, find it cheaper to get one from k on l, which he can exchange for one from l on i.

Illustrative Problem

This relationship is readily understood by considering an example of the three centers of London, Paris, and New York. Let the value of the dollar in francs be C_{ik}; the value of the dollar in pounds, C_{lk}; and the franc value of the pound, C_{il}. If the franc is worth 2.5 cents in New York, and if the pound is worth $4.66 in New York and 175 francs in London, we have the following:

$$C_{ik} = \frac{\$1}{0.025} \qquad C_{il} = 175 \qquad C_{lk} = \frac{1}{4.66}.$$

$$\frac{1}{0.025} \text{ should equal } 175 \times \frac{1}{4.66}.$$

If C_{ik} is less than the product of C_{il} and C_{lk}, it would be advantageous, in the exchange of dollars into francs, to buy a draft on London, and with this buy a draft on Paris. Conversely, if C_{ik} were greater than the product of C_{il} and C_{lk}, it would pay a trader in France to buy dollars in New York through a London draft.

The relationships among the different currencies are significant, for they explain why, in the movement of gold among three financial centers, there will always be two between which there will be no direct shipments. Between these two, which are not always the same, the rate of exchange will stay within the gold points. One other significant fact which these relationships emphasize is the fact that arbitragers, realizing the relationship which should exist among the various currencies, constantly watch for deviations from which they can profit. Their actions constantly tend to re-establish the ratio,

$$C_{ik} = C_{il} \cdot C_{lk}.$$

INTEREST ARBITRAGE

In the years immediately preceding the war, large sums of money were kept in very liquid form to be transferred quickly from one financial center to another to take advantage of the differences in interest rates among the different countries. These large transfers created serious financial problems.

One may, however, take advantage of higher interest rates in one specific country without assuming any risk due to fluctuating exchange rates. Suppose that the interest rate in New York on prime commercial paper is 3 per cent and that the rate on similar paper in Paris is only 2 per cent. A French banker could earn 1 per cent more on his funds in New York. If he simply transferred funds to New York, he might find that, when he wanted to transfer his funds back to Paris, the quotation in New York had increased and so he might lose more than he gained. The banker in Paris can obviate this difficulty, however, by using the futures market in the following manner:

May 1—Purchase in spot market $100,000 to be invested in New York at 3 per cent.

May 1—Sell $100,000 for 90-day future delivery.

August 1—Collect interest for 90 days on the $100,000 invested in New York.

August 1—Deliver the $100,000 sold in the futures market May 1, and collect the equivalent in francs.

The banker then has his funds back in francs, he has collected his interest at the higher rate, and he has not been subjected to the dangers of fluctuating exchange rates. Such a transaction would be profitable only if the discount on the forward francs and the expense involved totalled less than 1 per cent. If the forward rate on francs is at a premium, the banker making the interest arbitrage profits by this additional amount.

Such a situation is illustrated by the following example:

In London, in September, 1936, the rate of exchange on New York was $5.055, and the 90-day futures rate was $5.03; the interest rate in New York on 90-day bills was 0.25 per cent and in London was 0.53 per cent. Thus, the interest rate was higher in London than in New York, but an English banker could buy spot dollars, and sell dollars 90 days forward, then invest the dollars in New York at a lower rate of interest, and still make a profit. A similar situation existed between the franc and the dollar during most of 1936, when the premium on forward dollars was so great that the French banks were making from 15 to 40 per cent per annum by buying spot dollars and selling futures.

Sometimes, however, the futures rates are not such that profits can be made. An example of this is found under the conditions prevailing in January, 1936:

	New York	London
Interest on prime commercial 90-day paper	0.12 per cent	0.60 per cent

Sterling in the spot market was selling at $4.96 and 90-day futures were quoted at a discount of 1, or $4.95. This being so, it would not be profitable to make an interest arbitrage, since the loss on the future sale of sterling would more than offset the gain on the interest differential.

GOLD ARBITRAGE

A third kind of arbitrage may be called gold arbitrage. In the earlier discussion of the mint par of exchange (p. 1127), we saw that, when countries are on the gold standard, the rate of exchange normally will vary only within what was called the *gold points*—points established by the cost of shipping gold. This is true because bankers will carry on arbitrage in gold which will keep the rates within these limits. This operation is illustrated in the following example:

Assume that, before 1931, gold at the Treasury was sold at $20.67 an ounce and that it cost about 2 cents per pound sterling to ship gold to London. When the rate of exchange on London is $4.8665, it does not pay a banker to ship gold, but, when the rate rises above $4.8865 (4.8665+2 cents), the banker can ship gold and sell exchange against gold so shipped. In this way, the banker can get a profit without taking any risk at all, and the exchange rate is kept within the gold points.

When countries are not on a true gold standard, this kind of gold arbitrage is no longer possible. There is another type, however, which is dependent on the existence of a gold market in at least two countries with some freedom in the flow of gold. If gold may flow freely both ways, an arbitrage may occur in either direction. In the United States, however, the Treasury will buy gold but will sell it only under certain circumstances. The United States Treasury buys gold at $35 an ounce, less $\frac{1}{4}$ per cent, making a net price of $34.9125 per fine ounce.

If the rate for demand bills on Great Britain is $4.65, and the price of gold in London is £7 8*s*. 6*d*., an arbitrager finds it to his advantage to buy demand sterling at $4.65, exchange it for gold in London at £7 8*s*. 6*d*., pay the shipping cost of about 24 cents an ounce, and then sell the gold to the Treasury at $34.9125. To buy gold in London at £7 8*s*. 6*d*. when the exchange rate is $4.65, is the equivalent of paying $34.52625 an ounce, since

$$7.425 \times \$4.65 = \$34.52625.$$

To this must be added the shipping costs, making the total cost about $34.766 an ounce. Since the Treasury pays $34.9125, the net profit amounts to 14.6 cents an ounce.

Selling price ($35−0.25 per cent) =		$34.9125
Less:		
Cost (£7.425×4.65)	$34.52625	
Shipping costs	0.24	34.7663
Net profit per ounce		$0.1462

Much of the gold which flowed into the United States from 1937 to 1939 came in as the result of such transactions. It should be pointed

out that the gold inflow into the United States may be explained partially by the arbitrage of gold from the world gold market to the United States Treasury, but that a gold arbitrage may not be made in the opposite direction—*i.e.*, from the United States—unless the Treasury licenses gold exports. It should be noticed that little, if any, risk is involved in such operations, for, in any true arbitrage, the arbitrager is not speculating but is taking a simultaneous position in two markets where he can calculate his profits beforehand.

TEST YOUR KNOWLEDGE OF INTERNATIONAL MONETARY RELATIONS

26 If the cable rate is \$4.46375 and the discount rate is 4 per cent, the stamp tax $\frac{1}{20}$ per cent, and the commission $\frac{1}{4}$ per cent, (a) what is the 60-day rate, allowing a 3-day period of grace and using 365 days as a basis for calculating interest, and (b) what is the rate, using 360 days as a basis for calculating interest?

27 If you strongly suspected in September, 1931, that England would soon leave the gold standard, how could you have profited? The rates in New York on London were as follows:

Date	Spot Rate	Forward Rates	
		One Month	Three Months
September 19, 1931	4.86	$-\frac{9}{16}$	$-2\frac{1}{8}$
September 26, 1931	3.75	$-2\frac{1}{2}$	$-6\frac{1}{2}$
October 17, 1931	3.87	+1	+2

28 How could a speculator have profited, had he anticipated the following changes?

Date	Spot Rate	Forward Rates	
		One Month	Three Months
December 7, 1935	$4.92\frac{1}{2}$	$-\frac{3}{8}$	$-1\frac{1}{8}$
March 7, 1936	4.99	$-\frac{1}{4}$	$-\frac{3}{4}$
April 4, 1936	4.94	$-\frac{1}{4}$	$-\frac{3}{4}$
July 6, 1936	$5.02\frac{1}{4}$	$-\frac{3}{8}$	$-1\frac{5}{16}$

29 Fill in the following table for Italian lire, pounds sterling, and dollars:

£	92	$92\frac{1}{4}$	$92\frac{1}{2}$	$92\frac{3}{4}$	93
	\$1	\$1	\$1	\$1	\$1
4.65	....	....	....	....	....
$4.65\frac{1}{4}$	....	....	....	....	....
$4.65\frac{1}{2}$	....	....	....	....	....
$4.65\frac{3}{4}$	....	....	....	....	....

30 If you had to make payment in Rome of 250,000 lire when the quotation in New York on the lira was \$0.0505 and on the pound was \$4.65, and when the lira was quoted in London at $92\frac{1}{4}$, what would be the most profitable method of payment?

31 Assume that the Treasury price of gold is \$35 an ounce less $\frac{1}{2}$ per cent handling charge, and that the cost of shipping gold is 24 cents an ounce. The demand rate of sterling exchange is \$4.70, the price of gold in London is 146*s.* 10*d.* With only \$10,000, how much profit can be made in a gold arbitrage?

MATHEMATICS OF THE SECURITIES MARKETS

You have previously learned that bonds are securities issued by corporations as evidence of long-term indebtedness. Under the terms of the contract expressed in the bond and in a supplementary agreement called the *bond indenture*, the corporation agrees to pay the holder or owner periodic payments of interest at a fixed rate, together with a definite sum of money, called the *par*, or *face*, *value*, at a definite future date. The most common denomination for bonds is $1,000, but smaller denominations of $500 and larger denominations of $5,000 to $100,000 are not uncommon. Occasionally, $100 bonds are issued.

The interest rate named in the bond is called the *nominal*, or *coupon*, *rate*. A bond purchased at a price above par is said to be purchased at a *premium*, and the rate of interest received on the amount invested is less than the nominal rate. We speak of the rate received by the purchaser as the *effective rate*, or *yield*. If a bond is purchased for less than par value, we say that it is purchased at a *discount*, and the effective rate of interest will exceed the nominal rate. There may be some relationship between the risk involved in a bond and the price of the bond; but, by and large, the mathematical relationship is not easily determined, and indeed may not even exist. In the market at any given time, many good bonds may be selling at a discount and many poor bonds may be selling at a premium. In other words, the market price is not always a good indication of the intrinsic worth of a bond.

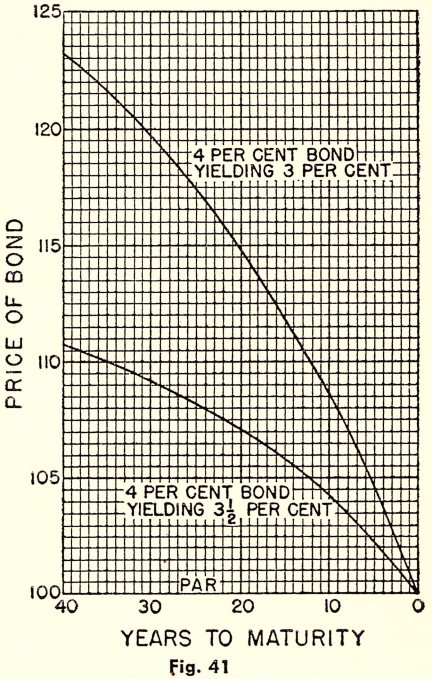


Fig. 41

Price Curves of a 4 Per Cent Bond with Constant Yields at Ten-year Intervals to Maturity

Evaluation of bonds

When a bond is bought, the buyer in effect gives up present goods—or rather, gives up claims on present goods—for claims on future goods. The future goods are represented by

a the principal, which he hopes will be repaid when the bond matures;
b the periodic interest payments which he hopes to receive during the life of the bond.

The calculation of the investment value, or purchase price, of the bond then amounts to the calculation of the present value of the face of the bond and the present value of the series of interest payments.

From the study of annuities, you have learned that:

C = maturity value of bond;
R = periodic interest, or dividend payment per period;
g = the ratio of total interest paid per year to the redemption price, C;
i = effective interest rate per period;
n = number of periods to maturity;
p = number of interest payments a year.

When the interest on a bond is payable p times a year, the interest is

$$R=\frac{Cg}{p},$$

and is payable p times a year for n years. The interest payments are thus an annuity of Cg payable in n installments. Therefore, the present value of the dividends is

$$\frac{Cg}{p}a_{\overline{n}|i}.$$

As you have previously learned, the present value of the redemption price is $C(1+i)^{-n}$. Thus, the purchase price of a bond, to yield an effective rate i, would be:

$$P=C(1+i)^{-n}+Ra_{\overline{n}|i}.$$

In order to facilitate the calculation of bond yields, several bond tables have been published. Samples of types in common use are represented in Tables XCVI and XCVII (pp. 1180 and 1181).

The Sprague Table (Table XCVI) shows the value to the nearest cent of a bond for $1,000,000 bearing interest from 3 to 7 per cent and yielding from $1\frac{1}{4}$ to 10 per cent. The table is divided into three parts dealing with low rates ($1\frac{1}{4}$ to $2\frac{1}{2}$ per cent), medium rates ($2\frac{1}{2}$ to 5 per cent), and high rates (5 to 10 per cent). The tables are made up for bonds on which interest is paid semi-annually, but additional tables are included which make it easy to work out values for bonds whose interest is payable quarterly or annually.

Illustrative Problem

An investor has purchased a 5 per cent bond at 101 with 14 years to run until it reaches maturity, and with interest paid semi-annually. He wants to find the yield of the bond by using the Sprague Table.

a Turn to the page dealing with 5 per cent bonds, interest payable semi-annually. (A section of this table is reproduced on page 1180.)

b Look down the *14*-year column for a bond priced at approximately 101.

In this example, the price of 1,010,045.56 furnishes a yield (according to the first column of the table) of 4.90. Since this price is as near to 101 as is necessary for ordinary purposes, the investor can accept the figure of 4.9 as the yield on the bond.

Suppose, however, that the price paid for the bond is not one given in the table. Then it is necessary to find the yield by interpolation. Thus, an investor wishing to find the yield on a 5 per cent bond due in 13 years and selling at 105 proceeds as follows:

He consults the tables to find the following prices and yields:

Price	Yield
$1,053,849.78	4.45
1,048,807.78	4.50

The difference in price is $5,042; the difference in yield, 0.05. Thus, a difference of $5,042 in the price of a $1,000,000 bond makes a difference of 0.05 in yield.

The price of

$$105(1{,}050{,}000-1{,}048{,}807.78=1{,}192.22)$$

is $\frac{1{,}192.22}{5{,}042}$ of the difference between the two prices given. By interpolation, the corrected yield for the bond selling at 105 is

$$4.50-\left(\frac{1{,}192}{5{,}042}\times 0.05\right)=4.488.$$

This is the common method of interpolation used for the Sprague or any similar table.

INTERPOLATION FOR INTERVALS

Another fact which one soon notices in finding yields on bonds is that the remaining life of a bond is not always a multiple of a half-yearly period, which is the interval shown in most bond tables.

Such tables will satisfy the ordinary investor, but when large sums are involved, more complete tables should be used. To meet the needs of institutions and large investors, the Financial Publishing Company has recently published a more complete table, called *Monthly Bond Value Table*, which is especially convenient to have when many yields must be calculated. For most persons, however, a table with fewer details will suffice if the procedure illustrated in the following example is used:

A 5 per cent bond with interest payable semi-annually which matures on July 1, 1959, is bought on April 16, 1944, at $102\frac{3}{4}$. What is the yield?

The life of the bond is not 15 nor $15\frac{1}{2}$ years, but is 15 years, 2 months, and 15 days. In this case, it is necessary to compute the yields of bonds both for 15 years and for $15\frac{1}{2}$ years, and then by interpolation find the yield on the particular bond running for 15 years, 2 months, and 15 days.

From the Sprague Table, we find the following figures for 15-year bonds:

Price	Yield
$1,032,032.65	4.70
1,026,604.29	4.75

Thus, for a price difference of $5,428.36, the difference in yield is 0.05. The difference between the price of $1,032,032.65 and the price paid of $1,027,500 is 4,532.65. If the yield at a price of $1,032,032.65 is 4.70, then, at the lower

price of $102\frac{3}{4}$, the yield would be higher, or 4.74175 calculated as follows:

$$\frac{4{,}532.65}{5{,}428.36}\times 0.05 = 0.04175.$$

$$4.70 + 0.04175 = 4.74175,$$

Interpolating to find the yield for $15\frac{1}{2}$ years, we find the following:

Price	Yield
$1,032,762.73	4.70
1,027,208.10	4.75

Thus, for a price difference of $5,554.63, there is a difference in yield of 0.05. The difference between the price of $1,032,762.73 and the price paid of $1,027,500 is $5,262.73. If the yield at a price of $1,032,762.73 is 4.70, then, at the lower price of $102\frac{3}{4}$, the yield would be higher.

$$\frac{5{,}262.73}{5{,}554.63}\times 0.05 = 0.0474.$$

$$4.70 + 0.0474 = 4.7474,$$

the yield for a bond bought at $102\frac{3}{4}$ with a $15\frac{1}{2}$-year maturity.

Since a 15-year bond has a yield of 4.74175 and a $15\frac{1}{2}$-year bond has a yield of 4.7474, the difference in yield is 0.00565. Six months is equivalent to 180 days. Two months and 15 days is the equivalent of 75 days.

$$\frac{75}{180}\times 0.00565 = 0.00235.$$

Yield for 15-year bond	4.74175
Add yield for 75 days	0.00235
Yield for 5 per cent bond running 15 years, 2 months, and 15 days	4.7441

One page from the Equitable Trust Company Rapid Bond Tables is shown on page 1181 (Table XCVII). These tables are constructed for bonds having interest rates of 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, 6, and 7 per cent. The tables show maturities from 6 months to 50 years in 6-months' intervals. The page reproduced is for bonds with 20 years to run to maturity. A similar page is shown in the complete table for each period from $\frac{1}{2}$ year to 50 years. It should be noted that the column at the left shows the yield per annum, the column headings show the coupon rate of the bond, and the body of the table shows the price of the bond. It will be noted that each basic price is followed by the decimal difference for one month, and also the decimal difference for one day, between each two 6-months' periods. This is done to facilitate the computation of bond prices and yields for bonds having other than an even number of interest intervals to run.

Illustrative Problem

At what price will a 3 per cent bond having 20 years, 5 months, and 20 days to run sell to yield 5.2 per cent?

Price for a 20-year bond		72.8462
5 months @ 0.064000 per month	0.320	
20 days @ 0.002133 per day	0.04266	
Deduct		0.36266
Price		72.48354

In the case of a premium bond, the difference would be added instead of subtracted. Thus, the price of a 5 per cent bond selling to yield 4.5 per cent for 20 years, 3 months, and 5 days would be computed thus:

Price for a 20-year bond, 5 per cent, selling to yield 4.5 per cent		106.5484
3 months @ 0.016733 per month	0.050199	
5 days @ 0.000557 per day	0.002785	
Add		0.052984
Price		106.601384

SIMPLIFIED METHOD FOR BONDS SELLING AT A PREMIUM

To be of much value, bond tables must necessarily be long and detailed; and since a small investor uses them infrequently, he does not feel that their purchase is warranted. Investment houses generally furnish bond tables to their clients, or give them the information they want. Often, however, investors do not have access to bond tables, or, if they do, the tables may not be detailed enough to give the necessary information.

All necessary calculations can be made by using the formulas above if annuity tables and compound-interest tables are available. This method will be the one normally used. If, however, no bond tables are available, it is not difficult to calculate either the price of a bond for a given yield, or to find the yield for a given price. The simplified method explained in the following paragraph gives results which are approximately accurate.[1]

Find the yield on Central Maine Power Company $3\frac{1}{2}$ per cent bonds due December 1, 1958, purchased in 1943 at 109; interest payable June 1 and December 1.

The purchase price of 109 means that the bonds sell for \$109 for each \$100 par value. Thus, a \$1,000 bond sells at \$1,090; a \$500 bond, at \$545; etc.[2]

[1] It should be repeated, however, that rule-of-thumb methods are not to replace the bond valuation tables, but merely to substitute for them under particular circumstances.

[2] Bond prices are almost always quoted in this way. The important exception is the quotation of United States Government bonds, which are quoted in 32ds of a point. Thus, a bond quoted at $100\frac{16}{32}$ might appear in the daily newspaper quotation as 100.16, indicated as a decimal but actually meaning $\frac{16}{32}$, which is accurately stated in decimals as 100.5.

By using the short method which we are illustrating, we should calculate the yield on the Central Maine Power bonds as follows:

Cost of the bond	109
Less: Redemption price (*i.e.*, maturity value)	−100
Premium	9
Years to maturity = 15	
Annual amortization (9 ÷ 15) = 0.6	
Annual interest	3.50
Less: Annual amortization of premium	−0.60
Net income	2.90

a $$\frac{\text{net income}}{\text{cost}} = \frac{2.90}{109} = 2.66 = \text{approximate yield.}$$

b $$\frac{\text{net income}}{\text{redemption price} + \text{annual amortization}} = \frac{2.90}{100.6} = 2.88.$$

c Averaging the two yields of steps a and b,

$$\frac{2.66 + 2.88}{2} = 2.77,$$

approximate yield by short method.

For premium bonds maturing in less than 50 years, an even more simplified method can be used with a high degree of accuracy. The formula is as follows:

$$\frac{\text{coupon} - \dfrac{\text{premium}}{\text{years to maturity}}}{100 + (0.6)\ (\text{premium})} = \frac{3.50 - \dfrac{9}{15}}{100 + (0.6)\ (9)} = \frac{2.90}{105.4} = 2.75.*$$

Discount bonds

Find the yield on Seattle Gas Company 1st and Refunding 5's due October 1, 1954, purchased October 1, 1943, at 92. (Interest payable April and October.)

Redemption price of bond	100
Less: Purchase price	− 92
Discount	8
Years to maturity = 11	
Annual accumulation (8 ÷ 11) = 0.73	
Annual coupon	5.00
Add: Annual accumulation of discount	+0.73
Net income	5.73

a $$\frac{\text{net income}}{\text{cost}} = \frac{5.73}{92} = 6.228,$$

an approximate yield which is too high.

b $$\frac{\text{net income}}{\text{redemption price} - \text{annual accumulation}} = \frac{5.73}{100 - 0.73} = 5.772,$$

an approximate yield which is too low.

*According to a bond, a $3\frac{1}{2}$ per cent bond with 15 years to maturity would yield 2.75 if purchased at 109.168 This figure indicates that the method used is fairly accurate.

c Averaging the yields of steps a and b,

$$\frac{6.228+5.772}{2}=6.00,$$

an approximate yield which is fairly accurate.

A shorter formula for discount bonds is:

$$\frac{\text{coupon}+\dfrac{\text{discount}}{\text{years to maturity}}}{100-(0.6)\ (\text{discount})}=\text{yield}.$$

Substituting the figures for the Seattle Gas Company bonds in this formula, we have:

$$\frac{5.00+\dfrac{8}{11}}{100-(0.6)\ (8)}=\frac{5.73}{95.2}=6.019 \text{ per cent.}$$

The yields calculated by the two methods differ, and in this case the one obtained in the first method is more nearly accurate. (Remember that in either method the result is a scientific approximation.)

TEST YOUR KNOWLEDGE OF BOND PRICES AND YIELDS

32 A 5 per cent bond on which interest is payable in April and October matures in October, 1957. The bond was bought April, 1944, at a price of 119. Use the Sprague Table to find the yield.

33 Assume that the Wabash, Chester, and Western Railroad Company has outstanding some 5 per cent bonds on which interest is paid semi-annually. If the bonds have 13 years to run to maturity and you buy one on the first day after an interest payment at 120, what yield will you receive on the bond, assuming (as the bond table does) that you plan to hold it to maturity?

34 A 5 per cent bond was purchased January 2, 1944, at 101. The bond matures October 1, 1957. What is the yield if interest is paid semi-annually?

35 Calculate the yield on a 5 per cent bond running 15 years, 6 months, and 15 days, purchased at $1,026,858. Interest paid semi-annually.

36 A 5 per cent bond due on August 1, 1960, sold at 111 on December, 1943. If interest is paid semi-annually, what is the yield on the bond?

37 Interest at 5 per cent is paid semi-annually on the bonds of a certain company. The bonds mature on January 1, 1959. If they were bought on June 1, 1943, at 106, what is the yield?

38 At what price will a 3 per cent bond which matures in 20 years, 5 months, and 10 days, sell today to yield 4.9 per cent? Use Equitable table.

39 At what price must a 5 per cent bond having 20 years, 2 months, and 20 days to run, sell to yield 4.7 per cent?

40 Find the yield to maturity on Detroit Edison General and Refunding 4's due October 1, 1965, bought at 112 in April, 1943. (Interest payable April and October.) Use the short method.

41 Using both methods, find the approximate yield to maturity of the following securities:

	Purchase Price	Issuer	Date of Purchase	Interest Rate	Maturity	Interest Paid
(a)	82	Standard Power & Light	8/1/43	6	2/1/58	February and August
(b)	92	Cleveland Union Terminal	4/1/43	5½	4/1/72	April and October

Calculation of interest coverage[1]

In the selection of desirable bonds for investment purposes, the investor may be aided materially by certain mathematical calculations. Perhaps the first calculation to be considered is that which is commonly referred to as the *interest coverage*—that is, the ratio of earnings available for bond interest to the amount required for bond interest.

If a corporation has only one bond issue outstanding with annual interest requirements of $4,000,000 and if the corporation has earnings of $12,000,000 available for bond-interest payments, it can be easily seen that the interest has been earned three times.

Assume, however, that the corporation has two bond issues outstanding, with interest requirements as follows:

First mortgage bonds	$ 400,000
First refunding bonds	200,000
Available for bond interest	1,200,000

There is available for the payment of interest on the first mortgage bonds $1,200,000, or three times the amount required for bond interest. After $400,000 has been deducted from $1,200,000, there is still available $800,000 for the payment of the $200,000 interest on the first refunding bonds. Since $\frac{\$800,000}{\$200,000}=4$, it can be said that the interest on the first refunding bonds is earned four times.

This method, which may be called the *prior deduction method* of calculating interest coverage, gives a very distorted picture of the true facts, and its use is apt to deceive a buyer into thinking that a junior mortgage bond or preferred stock is stronger than a prior lien or a first lien issue of the same company.

A second method of calculating interest coverage is called the *cumulative deduction method.* It is calculated as follows:

Available for bond interest		$1,200,000
Interest on first mortgage bonds	$400,000	
Times earned: $\frac{1,200,000}{400,000}=3$		
Interest on first refunding bonds	200,000	
		$ 600,000
Times earned: $\frac{1,200,000}{600,000}=2$		

In order to arrive at the coverage for the junior securities under this method, we combine the interest requirements and divide them into the total earnings available. The interest coverage would differ for the two issues. It would still be said that the first mortgage bond interest was earned three times on the average; but the interest on the refunding bonds would be described as having been earned twice rather than four times, as it was in the prior deduction method.

[1] The discussion of bonds in this section deals only with methods of calculating certain ratios. The interpretation and standards of ratios for such investments, outside the scope of this discussion, can be found in any good book on investments.

A third method—and the most desirable one—is called the *over-all* or *total deduction method.* Here the calculation is to see how many times *all* interest charges are earned. Using the figures from the foregoing example, we should calculate under the *over-all* method as follows:

$$\frac{\text{available for bond interest}}{\text{interest on first-mortgage bonds} + \text{interest on refunding bonds}} = \frac{1{,}200{,}000}{400{,}000+200{,}000} = 2.$$

Although the over-all basis for computing interest coverage is now most common, the other two methods are used often enough to make it imperative for an investor to examine carefully all statements of interest coverage and to verify them by his own computations.

Two other expressions are sometimes used to denote the interest coverage.

The first of these is the *factor of safety.*

If the interest is computed to be earned twice on the average, and if interest is considered to be equal to 100 per cent, then the factor of safety (which may be defined as the shrinkage necessary in earnings before bond interest cannot be met) is 100 per cent. If the interest coverage is $1\frac{3}{4}$ times (175 per cent), then the factor of safety is 75 per cent

(175 per cent − 100 per cent).

Less frequently used in investment literature, the second term, *margin of safety*, is defined as the ratio of the balance, after interest, to the earnings available for interest. Thus, if the interest is earned twice on the average, the margin of safety becomes

$$1 \div 2 = 50 \text{ per cent.}$$

If interest is covered $1\frac{3}{4}$ times, the margin of safety becomes

$$\frac{3}{4} \div 1\frac{3}{4} = 42\frac{6}{7} \text{ per cent.}$$

Relation of bonded debt to assets

In selecting bonds, a careful investor always considers the relationship of the debt to the net asset value. Consider a corporation with the following balance sheet:

Assets	$1,200,000	Current liabilities	$ 200,000
		Bonds	600,000
		Common stock	300,000
		Surplus	100,000
	$1,200,000		$1,200,000

Total assets	$1,200,000
Less: Current liabilities	200,000
Net assets	$1,000,000

If the $1,000,000 is considered the net asset value, the bonded debt is equal to $66\frac{2}{3}$ per cent $\left(\frac{600{,}000}{1{,}000{,}000} = 66\frac{2}{3} \text{ per cent}\right)$ of net assets.

A somewhat similar ratio (called the *stock equity ratio*) based on the stock capitalization, is found[1] by dividing the equity of the stockholders (in this case, common stock of $300,000 plus surplus of $100,000 = $400,000) by the bonded debt.

$$\frac{400,000}{600,000} = \frac{2}{3} = 66\frac{2}{3} \text{ per cent.}$$

This is equivalent to dividing the net assets by the bonded debt and deducting 1 from the quotient.

$$\frac{1,000,000}{600,000} = 1\frac{2}{3} \qquad 1\frac{2}{3} - 1 = \frac{2}{3}$$

Coming into more common use is the *stock value ratio*, which shows the relation of the stockholders' equity to bonded debt. It is calculated as follows:

1 = par value of bonds (*i.e.*, total bonds outstanding)
2 = number of preferred shares × market price per share
3 = number of common shares × market price per share

$$\text{stock value ratio for bonds} = \frac{2+3}{1}$$

$$\text{stock value ratio for preferred stock} = \frac{3}{1+2}$$

Convertible bonds

Often corporations include in their bond contracts a provision which permits a bondholder to exchange his bond at any time for a stipulated number of common shares. A security containing such a provision, whether it be a bond or a preferred stock, is called a convertible security. The ratio at which one security is exchangeable, or convertible, into another is known as the *conversion ratio*. There is a great lack of uniformity in the method of stating conversion ratios. For the sake of uniformity, this text will always describe conversion exchanges in the number of shares of stock into which a $1,000 bond is made exchangeable, or the number of shares of common stock into which one preferred stock may be exchanged.

CALCULATION OF THE CONVERSION RATIO

The Great Northern 4's of 1946 are convertible into preferred stock at $40. The conversion ratio is then:

$$\frac{\text{par value of bond}}{\text{value per share of stock}} = \frac{\$1,000}{\$40} = 25 \text{ shares.}$$

A $1,000 bond would be convertible into 25 shares of Great Northern Preferred stock.

Often the conversion ratio is stated in the bond; under such cir-

[1] In New York, certain types of bonds are not eligible for investment by savings banks unless they meet certain standards. In the procedure defined by the New York statute for making the preceding calculation, the surplus is not included as a part of the stock capitalization; therefore, such a ratio may be shown at a figure different from that which your own calculation shows. Notwithstanding the New York statute, the method which includes the surplus as part of the stock capitalization is preferable.

cumstances, the problem becomes one of stating the conversion ratio in the value of the stock. An example of this kind is as follows:

Carrier Corporation, Convertible $4\frac{1}{2}$ per cent bonds of 1948, are convertible into 36 shares of common stock through October 1, 1944, and into 32 shares between October 1, 1944, and July 15, 1948. In the first instance, the stock is therefore given an exchange value of $27.78 per share in conversion.

$$\frac{1{,}000}{\$36} = \$27.78.$$

Once a bondholder knows the conversion ratio, how is he to decide when to convert? In reaching his decision, he must consider the market price of the common stock, the market price of his bond, the income after conversion, as well as the income he is receiving on the bond. In many instances, an investor is not concerned about converting, but instead elects to sell his bond, because by converting he would change his position from that occupied by a bondholder to that occupied by a common stockholder, but the person who trades actively on the exchange seeks opportunity for profits, and finds them occasionally in convertible issues.

In 1937, Baldwin Locomotive Works issued some 6 per cent refunding mortgage convertible bonds due in 1960. These bonds were paid off by the company in August, 1943. Under the conversion feature of the bonds, it was provided that until September 1, 1945, the bonds could be converted at the option of the holder into 65 shares of common stock for each $1,000 bond; thereafter they were convertible into only 55 shares. Before September 1, 1945, the conversion parity was: $1,000 bond equals 65 shares of stock. The bond at par was convertible into 65 shares, or the price equivalent of

$$\frac{1{,}000}{65} = \$15.38.$$

Stocks and bonds fluctuate daily in price, and although the conversion parity of the stock was $15.38 when the bond was selling at par, a reduction in the price of the bond to 65, with the price of the stock above 10, would make conversion profitable, and the price of the bond would be likely to rise to the conversion level. It is usually true that when two securities are near conversion parities, whether the bond is selling above or below par, the fluctuations in the price of the stock will exercise a considerable influence on the price of the bond. It is possible to work out a whole series of conversion parities; and if the trader anticipates profits from dealing in convertible securities, he works out such a schedule so that he can see at a glance when the conversion privilege becomes valuable.

Between 1937 and 1943, the price of the Baldwin Locomotive Works bonds fluctuated from a low of 57 to a high of 139. Knowing the range of probable fluctuations, a trader would work out a schedule as follows: If the bonds were quoted at $55\frac{1}{4}$ ($552.50 for a $1,000 bond), and if one bond was convertible

TABLE c
CONVERSION PARITIES

Schedule of conversion parities, Baldwin Locomotive Works 6% debentures, due 1960, into 65 shares of common stock

Stock prices with decimals are approximations. When bond prices are in exact relationship, the stock price is not shown as a decimal. Thus, when the stock price is shown as 8⅛ or 9, the relation is exact, but when the stock price is shown as 8.5 or 9.1, the relationship is only approximately correct.

Bond Prices	Stock Prices	Bond Prices	Stock Prices	Bond Prices	Stock Prices	Bond Prices	Stock Prices
55	8.5	78	12	101	15.5	125	19.2
55¼	8½	79	12.1	102	15.7	125⅛	19¼
56	8.6	79⅝	12¼	102⅜	15¾	126	19.4
56⅞	8¾	80	12.3	103	15.8	126¾	19½
57	8.8	81	12.5	104	16	127	19.5
58	8.9	81¼	12½	105	16.1	128	19.7
58½	9	82	12.6	105⅝	16¼	128⅜	19¾
59	9.1	82⅞	12¾	106	16.3	129	19.8
60	9.2	83	12.8	107	16.5	130	20
60⅛	9¼	84	12.9	107¼	16½	131	20.1
61	9.4	84½	13	108	16.6	131⅝	20¼
61¾	9½	85	13.1	108⅞	16¾	132	20.3
62	9.5	86	13.2	109	16.8	133	20.5
63	9.7	86⅛	13¼	110	16.9	133¼	20½
63⅜	9¾	87	13.4	110½	17	134	20.6
64	9.8	87¾	13½	111	17.1	134⅞	20¾
65	10	88	13.5	112	17.2	135	20.8
66	10.2	89	13.7	112⅛	17¼	136	20.9
66⅝	10¼	89⅜	13¾	113	17.4	136½	21
67	10.3	90	13.8	113¾	17½	137	21.1
68	10.5	91	14	114	17.5	138	21.2
68¼	10½	92	14.1	115	17.7	138⅛	21¼
69	10.6	92⅝	14⅛	115⅜	17¾	139	21.4
69⅞	10¾	93	14.3	116	17.8	139¾	21½
70	10.8	94	14.5	117	18	140	21.5
71	10.9	94¼	14½	118	18.2	141	21.7
71½	11	95	14.6	118⅝	18¼	141⅜	21¾
72	11.1	95⅞	14¾	119	18.3	142	21.8
73	11.2	96	14.8	120	18.5	143	22
73⅛	11¼	97	14.9	120¼	18½	144	22.1
74	11.4	97⅛	15	121	18.6	144⅝	22¼
74¾	11½	98	15.1	121⅞	18¾	145	22.3
75	11.5	99	15.2	122	18.8	146	22.4
76	11.7	99½	15¼	123	18.9	146¼	22½
76⅜	11¾	100	15.4	123½	19	147	22.6
77	11.8	100¾	15½	124	19.1	147⅞	22¾

into 65 shares of common stock, the equivalent value of common stock in conversion would be

$$\frac{\$552.50}{65 \text{ shares}} = 8\frac{1}{2}.$$

If the bonds were selling at 104 and were convertible into 65 shares, then in conversion each common share would have a value equivalent to

$$\frac{\$1{,}040}{65 \text{ shares}} = 16.$$

In this way, a table such as Table c could be worked out.

No general conversion tables are available. They are easily constructed and are generally developed to fit each particular situation. From Table c, an investor can see if it would pay him to buy Baldwin Locomotive bonds and sell stock if the bonds were selling at $99\frac{1}{2}$ and the stock at $15\frac{1}{2}$. The table shows that, with bonds at $99\frac{1}{2}$, the common stock would have to have a value of more than $15\frac{1}{4}$ a share to make conversion profitable.

If the price of the bond should fall to $84\frac{1}{2}$, at what price would the common stock have to sell to make conversion profitable?

The table of bond prices of $84\frac{1}{2}$ shows that the conversion parity for stock is 13. Thus, the stock would have to sell above 13 if buying a bond and converting it were to be profitable.

A conversion schedule is developed by using the ratio of

$$\frac{\text{bond prices}}{\text{number of shares}}.$$

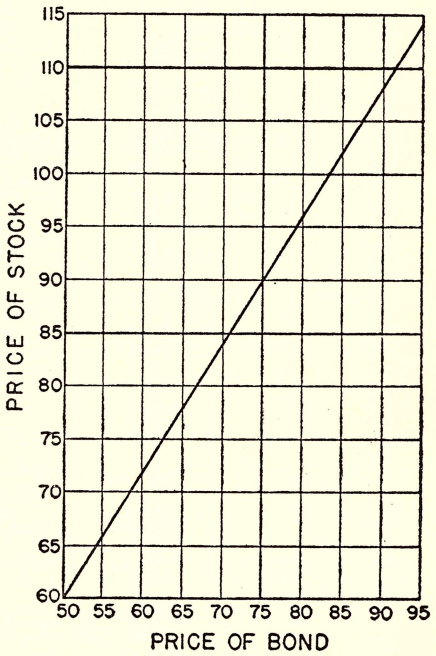


Fig. 42

Chart Showing Conversion Parities of Bond and Stock Prices for $1,000 Bond Convertible into $8\frac{1}{3}$ Shares of Common Stock

(The line represents the conversion parity. If the price of the stock is above the conversion parity line, conversion is profitable.)

If the price of a bond is divided by the conversion ratio, the quotient is the parity price of the stock. We must remember, however, that, when bond prices are quoted, they do not represent the price at which the actual transfer takes place, since the price which must be paid is the quoted price plus accrued interest. This adjustment is always made at the time of sale (unless quotation is "flat", which means that bonds are sold at the quoted price with no adjustment for unpaid interest).

Since the quoted price of the bond does not show the accrued interest,

a table of conversion ratios based on quoted prices is not exactly accurate. The prices of stocks and bonds are not on a strictly comparable basis, since stocks are always sold flat—that is, the price quoted includes the accrued portions of the dividends. If accurate comparisons are to be made, the price of one or the other must be adjusted.

In the conversion tables which appear on page 1155, the price of the bond is the quoted price—which does not include accrued interest; and the parity price of the stocks was calculated by dividing the quoted bond prices by the conversion ratio. Since the market price of stocks is on a flat price basis, it is necessary to deduct the accrued dividends from the price of the stock before we can compare this latter price to the stated parity. We can say that the conversion point has been reached when the market price of the stock, less the accrued dividend, is equal to the parity price of the stock.

USING ACCRUED DIVIDEND TABLES

In order to find accumulated dividends with facility, we find it wise to construct, or consult, an *accrued dividend table*. This table can be rather simple since daily fluctuations on the New York stock market for stocks selling over 1 are quoted in terms of $\frac{1}{8}$ of a point. Thus, a table may be constructed to show the accumulated dividends only to the extent that the stock-exchange fluctuations would occur. Thus, in the case of a company paying dividends at the rate of 4 per cent a year, the dividends would accrue at the rate of 1 per cent a quarter; and, in order to amount to $\frac{1}{8}$ of a point in market price, the dividends would have to accumulate for approximately 11 days.

$\left(\frac{365}{4}=91 \text{ days to accrue 1 per cent; } \frac{91}{8}=11 \text{ days to accrue } \frac{1}{8} \text{ per cent}\right)$

If the stock were paying dividends at the rate of 2 per cent a year, the dividends would amount to only $\frac{1}{2}$ per cent per quarter, and it would take approximately 23 days for dividends to accumulate to $\frac{1}{8}$ per cent.

Table XCVIII, based on 365 days, shows the rate at which dividends accrue. The upper line, shown as the annual dividend rate, is expressed in percentages. The single column headed "Interest on $100" shows the accumulated dividends expressed in fractions equal to $\frac{1}{8}$ per cent. Thus, a $100 share of stock which pays 8 per cent has dividends accumulating at the rate of $\frac{1}{8}$ per cent in six days and 1 per cent in 46 days. A share of stock selling for $25 and paying 8 per cent would accumulate accrued dividends equivalent to 1 per cent per $100 in 46 days, or the equivalent of $\frac{1}{4}$ point per share.

The use of the table to determine if conversion is profitable is illustrated in the following:

Illustrative Example

If in 1943 the Phillips Petroleum bonds (mentioned in problem 46) sold at 110 and the stock at 50, 23 days after a common dividend date, is conversion profitable at the present prices, assuming the annual dividend rate to be \$2?

$$\frac{\$1{,}000}{45.50} = 21.978 \text{ shares of stock}$$

One bond is convertible into 21.978 shares of stock
Current dividend on stock = \$2 (4% on \$50)
Number of days since last dividend payment = 23
To find parity price of stock,

$$\frac{\text{price of bond}}{\text{conversion ratio}} = \frac{1{,}100}{21.978} = 50.05.$$

Look now to the accrued dividend table. Under the section for *4* per cent, we find that for *23* days, dividends equivalent to $\frac{1}{4}$ per cent have accrued. Therefore, if conversion is to be profitable, the price of the stock must be equal to, or greater than, the conversion parity after $\frac{1}{4}$ point for accrued dividends has been subtracted. Subtracting $\frac{1}{4}$ per cent from 50, we find that $49\frac{3}{4}$ is below the point at which conversion would be profitable.

In recent years, several convertible issues, such as Allis Chalmers 15-year convertible S.F. debentures 4 per cent due 1952, contain the provision that "adjustment will be made for accrued interest and dividends upon conversion". Under such circumstances, the type of calculation explained above would be made by the company, and the investor would make his decision to convert or not to convert on the basis of his conversion schedule without adjusting for accrued dividends. On the other hand, some bonds, such as Consolidated Textile Company income 15-year convertible 5's of 1953, specifically provide that "no adjustment or payment on account of any accrued or unpaid interest on the debentures will be made". Under such circumstances, the bond is accepted at face value; and, if the common stock pays regular dividends, adjustment would entail the *addition* of the accrued dividend on the stock to determine whether conversion is mathematically profitable.

Thus far, we have discussed the profitableness of conversion purely on the basis of the prices of the securities involved. Discrepancies will not long exist between the prices of two securities, one of which is convertible into the other. Arbitragers will soon prevent the senior security from selling below the conversion parity, since it is always

possible to buy bonds and convert them for delivery against short sales of stock. To illustrate this practice, assume the following situation:

A bond is convertible into 40 shares of common stock. If the bond is quoted at 100 and the stock is selling for 28, a trader would:

a	Sell 400* shares of stock at 28	$11,200
b	Buy 10 bonds @ $100 each	10,000
	Gross profits (ignoring commissions)	$ 1,200

He could convert the bonds into stock and deliver the stock he received to cover the stock he sold.

Another important item to consider, in deciding whether to convert, is the income to be derived from the securities involved. In determining conversion on the basis of income, the investor must weigh not only the yield on the two securities but also the degree of risk involved. The fact that a fixed relationship will tend to exist between the prices of the two securities, once the price of the common stock rises to conversion parity, makes it necessary to have some basis other than price on which to judge the desirability of conversion.

Assume, for example, that a 6 per cent preferred stock is convertible into common stock on a share-for-share basis. On the basis of income, it would be profitable to convert as soon as the common dividend exceeded the preferred dividend of 6 per cent, but the chance that the preferred dividend would continue in the future, while the dividend on the common might be discontinued, would prevent many people from converting. On a purely mathematical basis, one can compare only the yields. This, however, is rarely the determining factor.

TEST YOUR KNOWLEDGE OF MATHEMATICS OF SECURITIES

42 Compute the coverage of interest charges for each bond issue of the Pennsylvania-California Corporation by each of three methods. What is the factor of safety and what is the margin of safety under each computation?

Net earnings after depreciation and taxes	$1,000,000
Interest on first mortgage bonds	500,000
Balance	$ 500,000
Interest on first refunding bonds	100,000
Balance	$ 400,000
Interest on debenture bonds	200,000
Balance	$ 200,000
Dividends on preferred stock	50,000
Net available for common stock	$ 150,000

43 The annual dividend of a certain stock is $2 a share, and the shares are currently quoted at $50. If the dividend is paid quarterly and the last dividend was paid 46 days ago, what is the accrued dividend per share?

44 Assume that the X Manufacturing Company has bonds convertible into 65 shares of common stock. (Use the conversion parity table of the Baldwin Locomotive Works on page 1155.) If the common stock has an

* In this illustration, 400 shares is used simply as a convenient figure.

annual dividend of $1.74 and if the last payment was made 76 days ago, would it be profitable to convert if the bond is selling at 91 and the stock at $14\frac{1}{2}$?

45 On December 31, 1942, the United Biscuit Company had outstanding the following securities:

$3\frac{1}{2}$ per cent debentures	$6,580,000
25,000 shares of preferred stock, average market price of 109	
468,283 shares of common stock, average market price of 13	
Total assets	26,791,299
Current liabilities	3,736,168
Net available for bond interest	3,879,318
Interest on debentures	230,300

Calculate (a) the interest coverage, (b) the ratio of bonded debt to net assets, (c) the stock-value ratio for bonds, and (d) the stock-value ratio for preferred stock.

46 The Phillips Petroleum Convertible $1\frac{3}{4}$ per cent bonds due in 1951 are convertible into common at $\$45\frac{1}{2}$ to November 1, 1946, and at $\$50\frac{1}{2}$ thereafter. (a) What is the conversion ratio before November 1, 1946? (b) What is the conversion ratio after November 1, 1946?

47 United Aircraft 5 per cent cumulative preferred stock is convertible into common at $40 a share until January 1, 1952. Construct a conversion table for these two stocks. Look at the New York Stock Exchange quotations in the newspaper and see from your table if conversion would be profitable today at the present price.

48 Find the accrued dividends on a $100 par value share of stock which pays dividends at the rate of 6 per cent, two months after the last dividend was paid.

49 The annual dividend rate on a share of stock is $9. What are the accrued dividends per share 8 weeks after the last dividend?

Stock purchase warrants

A privilege often given to senior security holders, somewhat similar to the convertible feature, is known as an *option warrant* or stock purchase warrant. These privileges, which may run for a limited time or which may be perpetual, give the holder the right to buy stock from the company at a fixed price. For example, the Atlas Corporation has outstanding some stock purchase warrants, known as the Atlas Corporation Warrants, which give the right to buy common stock from the company at $25 per share. Such certificates are often actively traded on the New York Curb Exchange since under certain circumstances they may be extremely valuable.

Often the warrants permit payment to be made in some other security of the company. Thus, Electric Power and Light Warrants give the holder the right to buy common stock at $25 per share, but payment may be made either in cash or by tendering second preferred stock at $100 a share.

The value of such a warrant can be calculated as follows:

S = subscription price at which the option warrant holder may buy new stock.
M_c = market price per share of common stock outstanding.
V_w = exercisable value of option warrant.
M_w = market price of warrant.
P = market price of senior security (in this case, the preferred stock) which may be tendered in payment for the common stock.
r = ratio at which senior security will be accepted in lieu of cash in exercising the warrant.

In the example we are working,

$$S=\$25;\ M_c=8\tfrac{1}{2};\ M_w=4\tfrac{1}{4};\ P=16;\ r=\frac{\$100}{25}=4,$$

since, under the conditions stated, one share of preferred stock plus four warrants may be used to obtain four shares of common stock.

$$M_c-\frac{P}{r}=V_w.$$

Substituting, we have

$$8\tfrac{1}{2}-\frac{16}{4}=4\tfrac{1}{2},$$

the exercisable value of the option warrants. If the warrants were quoted in the market at $4\frac{1}{4}$, they could be profitably purchased, and exercised by using preferred stock to make the subscription payment. Such a condition can be only temporary, for arbitragers or traders, noticing the discrepancy, will soon buy warrants and preferred stock on the market, then will exercise the warrants, and sell the common stock profitably. Therefore, we can conclude that M_c-M_w will, with rare exceptions, be equal to or less than the subscription price, S, or the quotient of the market price of the senior security, P, divided by the ratio, r, at which it may be tendered in payment for the common $\frac{P}{r}$.

$$M_c-M_w\leqq\frac{P}{r}\text{ or }S.$$

Warrants are usually purchased in anticipation that one of the securities involved will move in such a way as to make it possible for the owner of the warrant to profit either from exercising or selling the warrant. It is therefore important to know how to determine what changes are necessary to make an operation successful. The market price of the common must rise to the point at which it is equal to the sum of the subscription price plus the cost of the warrant before the warrants can be exercised with no loss. When no other securities are involved, this is not a difficult calculation.

If another security may be used in lieu of cash in payment for the new stock, another factor may have to be considered, thus adding to the complexity of the problem. The relation of $\frac{P}{r}$ may also have to be considered, especially is P if less than the exchange value. These relationships can perhaps best be illustrated with a problem con-

cerning a speculator who has bought warrants of the Maniti Sugar Company, planning either to exercise or to sell his warrants at a profit.

Illustrative Problem

The Maniti Sugar Company has outstanding option purchase warrants which run until November 5, 1947. The holder of each warrant is given the right to subscribe to common stock at the rate of $12.50 for each warrant he possesses. The company's 4 per cent sinking-fund bond due in 1957 may be tendered in lieu of cash at the principal amount plus accrued interest for payments of the subscription price. In 1941, the bonds sold at 32, the warrants at 1, and the common at $4\frac{1}{8}$. What changes would need to take place in the price of these securities to make the purchase of the warrants at 1 a successful speculation?

$$S=\$12.50;\ M_c=4\tfrac{1}{8};\ M_w=1;\ P=32;\ r=\frac{100}{\$12.50}=8$$

$$M_c-\frac{P}{r}=V_w=\text{the exercisable value}$$

$$4\tfrac{1}{8}-\frac{32}{8}=\frac{1}{8}.$$

Here we see that, although the market price of the common is $8\frac{3}{8}$ points below the subscription price, the warrants still had an actual value because the senior security which could be tendered in lieu of cash was also selling below its exchange value. A speculator buying the warrant at 1, however, would be paying $\frac{7}{8}$ of a point more for each warrant than its exercisable value. Such a speculator must have believed that any one of several things might happen to give the warrant greater value. If the price of the common stock rises, he will gain.

Assume that the market price of the common rises to $5\frac{1}{8}$ and that the bond continues to sell at 32. What is the exercisable value of the warrant?

$$5\tfrac{1}{8}-\frac{32}{8}=1\tfrac{1}{8}.$$

If the price of the bond should fall, say to 28, and the price of the common stock should remain unchanged, the value of the warrant would be determined as follows:

$$5\tfrac{1}{8}-\frac{28}{8}=1\tfrac{5}{8}=\text{the exercisable value of the warrant.}$$

If the price of the common should rise, it is extremely likely that

the price of the bond would also rise; or, if the price of the bonds fell, the same forces which caused their decline would be likely to force down the price of the common. If both prices move in the same direction and proportionately, the warrants will not gain in exercisable value until the *bonds sell for par*. By early 1943, the bonds sold at 64, the common stock at $8\frac{1}{8}$, and the warrants were quoted at $1\frac{7}{8}$. Their exercisable value was $8\frac{1}{8}-\frac{64}{8}=\frac{1}{8}$.

If, under such circumstances, a rumor that the company was to declare a sizable dividend on common should bring about a rise on those shares to 10, while the bond prices remained unchanged, the warrant holders would profit. What factors might change in such a way that an increase in the price of common stock would result? In the first place, it must be remembered that the company has not paid a dividend on this stock. Therefore, until the prospects for dividends appear good, it is unlikely that the common will sell at a price sufficiently high relative to the bonds to warrant conversion. On the other hand, if it now appears that the earnings of the sugar company will be especially good in the post-war years before 1947 and that the price of the common will rise, the speculator has in the warrants an excellent medium for speculation.

Assume that the warrants were quoted at 2 when the common was selling at 8. If earnings increase, the price of the bonds will move toward par, as the common increases in price. When both are at par, the warrants will have no exercisable value, but speculators will force up the price of the warrants, since, when the common is selling at $12.50, every dollar increase in market price (at $12.50, an increase of $1 is an 8 per cent increase) of the common will increase the value of the warrant by $1. Thus, the percentage increase in price of the warrant is magnified. How far speculation would drive the price of the warrant cannot be determined exactly. If the stock continued to pay no dividends, many would feel justified in buying warrants rather than stock. For one thing, the loss of interest on the amount involved in the purchase of the warrants would be less than the loss of the interest involved in the purchase of the common stock. If the warrants did not then expire, the savings in interest would, in a reasonable time, justify the original higher purchase price.

John Burr Williams developed[1] the following formula, which enables one to calculate how many years n would have to lapse with no dividends to make the warrants and the common stock equally attractive at their present prices:

$$n=\frac{\log (M_c-M_w)-\log S}{\log v}.$$

[1] *The Theory of Investment Value*, Harvard University Press, Cambridge, Massachusetts, 1938, p. 546.

This formula would not apply to the Maniti Sugar Warrants, since they have a definite expiration date. Assume, however, that they were to run indefinitely, and that:

$$M_c = 15$$
$$S = 12\frac{1}{2}$$
$$M_w = 8$$
$$i = 6 \text{ per cent}$$
$$v = \frac{1}{1.06} = 0.9434.$$

Inserting these figures in the equation, we have:

$$n = \frac{\log (15-8) - \log 12.50}{\log 0.9434} = \frac{0.845098 - 1.09691}{9.974696 - 10} = 9.95.$$

Thus far, we have dealt with only a non-dividend-paying common stock. Suppose that the holder of the warrant has the right to subscribe to a stock paying a regular dividend. Then the relation between the price of the option warrant and the price of the common stock is more complex. When another security may be used in lieu of cash in exercising the warrant, an additional complicating factor is added, but, in any event, the need of the funds to exercise a warrant is a factor which varies in its influence on the spread between the two prices. Thus, exact accuracy cannot always be achieved. The dividend rate on the common stock, as well as the anticipation of changes in the dividend rate, will have an important bearing on the relationship of the price of the option warrants and the price of the common stock. We have already seen that

$$M_c - M_w \leqq S, \text{ or } \frac{P}{r}.$$

This equation expresses the maximum differences between the two prices. The minimum spread between the two will be governed by what the investor determines is a fair return on the money which he has to invest. The difference between M_c and M_w will tend never to be *less* than an amount on which the dividend affords a fair return. If it were less, warrant holders would sell their warrants and buy stock at the market price; thus, they would be able to obtain stock with the proceeds of the sale of the warrants, together with additional funds, at a price which gives a return better than one which they consider to be fair.

This explanation may be clarified by the following example: Assume that a stable company which pays a fixed dividend (R) of $3.60 a share, has warrants outstanding which give the holder the right to subscribe to stock at $70 a share. Assume, further, that, according to investors, a return of 6 per cent (i) is a fair one on such stock. Whenever the stock can be acquired at less than $\frac{\$3.60}{6 \text{ per cent}}\left(\frac{R}{i}\right) = \$60$, the investor

will obtain what is to him a better than average return. Consequently, the difference of $M_c - M_w$ must be greater than \$60, or the investor will sell his warrants, and with additional cash will buy stock of the company at the market price, since his return on the additional funds invested will be what he considers above normal.

In the example given, if the price of the stock should fall to 65 and if the warrant were quoted at 10

$$\left[(M_c - M_w) < \frac{R}{i} \text{ or } 65 - 10 = 55, \text{ which is less than } \frac{\$3.60}{6 \text{ per cent}} = \$60 \right]$$

the holder of warrants could sell them for \$10 each, and, by adding \$45 to this sum, could acquire a stock which furnished a yield of more than 6 per cent. Therefore, we can conclude that:

$$\frac{R}{i} \leqq (M_c - M_w), \text{ or } \frac{R}{M_c - M_w} \leqq i.$$

We have previously learned that $M_c - M_w \leqq S$ or $\frac{P}{r}$. A combination of these two expressions gives a *rational upper and lower limit* for the spread between warrants and dividend-paying common stock.

$$\frac{R}{i} \leqq (M_c - M_w) \leqq S \text{ or } \frac{P}{r}.$$

The limits of the maximum spread between the price of the common and the price of the warrants is more exactly determined and more effective in the market than is the determination of the minimum spread, for, in the latter instance, there are present such variable factors as the investor's judgment as to future dividend policies, managerial changes, the subjective concept of what is a fair return, and a host of others. It must also be emphasized that our discussion deals with the mathematics of investment values, and that market prices are often irrational. The successful investor, however, must make his decisions on a rational basis, and it is with that basis that we are here concerned.

If an investor has purchased an option warrant with the anticipation of exercising it—that is, not seeking merely a speculative rise in value—, he will hold it until the dividend on the common stock is large enough to justify the payment of the subscription price. In order, therefore, to determine the spread between M_c and M_w, we must find an approximation of n, the number of years until the warrants are to become exercisable. One who holds the warrants for n years and who keeps invested elsewhere an amount equal to the subscription price, at what he considers a fair rate of return, i, would receive the same return as one who holds the common shares for n years and who received the dividends. By formula, then:

$$M_w + Sv^{n\not p} = M_c - R\left(a_{\overline{n}|\frac{i}{i}}\right).$$

This is equivalent to saying that the cost of the stock to the warrant buyer is the cost of the warrant, plus the present value of his future subscription, and that the total outlay of the purchaser of common stock is the purchase price of the common less the present value of dividend payments for n years.

Transposing this formula, which gives the spread between the common stock and the warrants, we get

$$M_c - M_w = Sv^{np} + R\left(a\overline{\left|\frac{np}{i}\right.}\right).$$

Conversely, we can solve the same equation for n instead of for $(M_c - M_w)$, and determine the number of years which the *market* expects to elapse before the exercise of the warrants becomes justified. To solve for n, we should give the formula the form of:

$$n = \frac{\log\left[(M_c - M_w) - R\left(a\overline{\left|\frac{np}{i}\right.}\right)\right]}{p(\log Sv)}.$$

It may aid an investor to make a satisfactory investment if he realizes precisely how much time, on the basis of price, the market expects to elapse before the exercise of the warrants becomes justified. With this knowledge, he may be able to select satisfactory investments and to reject those which are likely to prove unsuccessful.

Stock rights

Warrants may be issued for any one of several reasons, and because of their teleological basis, they usually run for long periods, if not perpetually. Stock purchase rights, or stock rights, are similar to warrants in that they give the holder the privilege of subscribing to a fixed amount of stock at a definite price. They differ from warrants, however, in purpose and in length of life. The laws of many States provide that, when a corporation plans to expand through the sale of additional stock, the existing stockholders must be given a pre-emptive right to subscribe to the new stock before outsiders are given the opportunity. This privilege gives rise to stock rights, which, like warrants, are transferable certificates generally traded on the organized exchanges. Unlike warrants, these instruments generally have a life limited to a very short period of time. Their use and the calculation of their value are illustrated if we assume the following conditions:

The stock of the Dew Chemical Company has been selling at \$45 a share, and the old stockholders are given the right to subscribe to one new share at \$30 for each four old shares held. If an investor went into the market and purchased four old shares with the rights attached, the cost of the four shares would be \$180. For an additional \$30 and the rights from the four old shares, he could buy one new share. His total investment for five shares would therefore be \$210. Since he

now has five shares, the average price paid has been \$42 a share $\left(\frac{\$210}{5}=\$42\right)$. After the rights have been exercised and the new stock issued, each share is like every other share, and the market price of all shares would tend to drop to \$42. It is easy to see that the privilege of buying stock for \$30 which will immediately sell for \$42 is worth \$12.

$$\$42-\$30=\$12.$$

In order to subscribe to one new share of stock, he needed the rights from four old shares. Their value must be $\$3\left(\frac{\$12}{4}\right)$ each. The value of the right attaching to each old share is referred to as the New York Right. These rights are quoted on the New York Exchange on the basis of the value of the rights attaching to each old share. If it is necessary to have the right attaching to more than one share in order to subscribe to one new share, the value of the New York Right (*i.e.*, the right attaching to each old share) will be less than the value of a right to subscribe to one new share. The right to one new share is called a Philadelphia Right, to distinguish it from a New York Right.

The formula for calculating the theoretical value of a New York Right is as follows:

$$V=\frac{m-S}{r+1}$$

where V = the value of the right
m = market price per share of the old stock
S = cost per share of the new stock,—*i.e.*, subscription price
r = the number of old shares it is necessary to own in order to be given the right to subscribe to one new share.

Substituting our values from the preceding example, we have:

$$m=\$45;\ S=\$30;\ r=4;\ V=\frac{45-30}{4+1}=\frac{15}{5}=\$3.00.$$

The value of the Philadelphia Right—that is, the value of the right to purchase one new share—is equal to

$$rV=4\times\$3=\$12.$$

As soon as the rights are issued, the stock sells ex-rights—that is, purchasers of the stock do not receive the rights, since they are retained by the person owning the stock at the time the rights were issued. When stock is selling ex-rights, the value of a New York Right would be determined by the following formula, with M equalling the value of the stock ex-rights:

$$\frac{M-S}{r}. \qquad \text{I}$$

Substituting the figures from the preceding situation, we get

$$\frac{\$42-30}{4}=\frac{12}{4}=\$3.$$

The value of a Philadelphia Right would be determined by the formula:

$$M-S. \qquad \text{II}$$

Using the figures as in I, we get $\$42-\$30=\$12$.

The real problem involved, however, is the choice of whether to exercise the right or to sell it in the market. If the holder of the stock has purchased it on the basis of a specific yield, he must determine as best he can how the additional assets will affect the income and dividend policies of the company. Assume that a large New York bank whose stock sells at 80 has paid annual dividends of $4 for the past 20 years. If earnings have been high enough to give adequate protection to such a dividend, and if the stockholders are given the right to subscribe to one new share of stock at $50 for each 5 old shares, the theoretical value of the New York Rights would be

$$\frac{80-50}{5+1}=\$5$$

and the stock ex-rights would sell for $75. If, however, the holder of the stock expects that in the future the stock will sell at the same specific yield of 5 per cent as it has in the past

$$\left(\frac{\$4 \text{ annual dividend}}{\$80 \text{ market price}}=5 \text{ per cent}\right),$$

and if he expects dividends to continue in the future at $4 a share, then at the end of two years he would expect the stock of the bank again to be selling at 80. He would have gained by exercising his right rather than selling it. Had he sold the rights at $5 each, he would have gained $25. By exercising the rights, he bought for $50 a share of stock now selling for $80. During the interim he has received income on the stock, and the market has appreciated on all shares he owns.

TEST YOUR KNOWLEDGE OF STOCK RIGHTS AND WARRANTS

50 The warrants of the United Corporation give the holders the right to subscribe to common stock at $27.50 a share. On December 31, 1929, these warrants sold at $14\frac{7}{8}$ and the stock sold at $32\frac{1}{4}$. If the stock paid no dividend, how many years would need to elapse without dividends to justify the spread of only $17\frac{3}{8}$ points instead of $27\frac{1}{2}$ points between the warrants and the stock? Interest at 6 per cent.

51 When the stock of Eastern Airlines was quoted at 44, stockholders were given the right to subscribe to new shares at $32 a share, in the ratio of one new share for each four shares held. (a) Calculate the value of a New York Right. (b) Calculate the value of a Philadelphia Right.

52 When Montgomery Ward stock was selling at $50 a share, stockholders were given the right to subscribe to one new share at $40 for each $6\frac{3}{4}$ shares held. Calculate the value of (a) a New York; (b) a Philadelphia Right.

TABLE LXXXIX
FUTURE VALUE OF $1

$$s=(1+i)^n$$

n	$\frac{1}{2}\%$	$\frac{3}{4}\%$	1%	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
1	1.00500	1.00750	1.01000	1.01250	1.01500	1.01750
2	1.01003	1.01506	1.02010	1.02516	1.03023	1.03531
3	1.01508	1.02267	1.03030	1.03797	1.04568	1.05342
4	1.02015	1.03034	1.04060	1.05095	1.06136	1.07186
5	1.02525	1.03807	1.05101	1.06408	1.07728	1.09062
6	1.03038	1.04585	1.06152	1.07738	1.09344	1.10970
7	1.03553	1.05370	1.07214	1.09085	1.10985	1.12912
8	1.04071	1.06160	1.08286	1.10449	1.12649	1.14888
9	1.04591	1.06956	1.09369	1.11829	1.14339	1.16899
10	1.05114	1.07758	1.10462	1.13227	1.16054	1.18945
11	1.05640	1.08566	1.11567	1.14642	1.17795	1.21026
12	1.06168	1.09381	1.12683	1.16076	1.19562	1.23144
13	1.06699	1.10201	1.13809	1.17526	1.21355	1.25299
14	1.07232	1.11028	1.14947	1.18996	1.23176	1.27492
15	1.07768	1.11860	1.16097	1.20483	1.25023	1.29723
16	1.08307	1.12699	1.17258	1.21989	1.26899	1.31993
17	1.08849	1.13545	1.18430	1.23514	1.28802	1.34303
18	1.09393	1.14396	1.19615	1.25058	1.30734	1.36653
19	1.09940	1.15254	1.20811	1.26621	1.32695	1.39045
20	1.10490	1.16118	1.22019	1.28204	1.34686	1.41478
21	1.11042	1.16989	1.23239	1.29806	1.36706	1.43954
22	1.11597	1.17867	1.24472	1.31429	1.38756	1.46473
23	1.12155	1.18751	1.25716	1.33072	1.40838	1.49036
24	1.12716	1.19641	1.26974	1.34735	1.42950	1.51644
25	1.13280	1.20539	1.28243	1.36419	1.45095	1.54298
26	1.13846	1.21443	1.29526	1.38125	1.47271	1.56998
27	1.14415	1.22354	1.30821	1.39851	1.49480	1.59746
28	1.14987	1.23271	1.32129	1.41599	1.51722	1.62541
29	1.15562	1.24196	1.33450	1.43369	1.53998	1.65386
30	1.16140	1.25127	1.34785	1.45161	1.56308	1.68280
31	1.16721	1.26066	1.36133	1.46976	1.58653	1.71225
32	1.17304	1.27011	1.37494	1.48813	1.61032	1.74221
33	1.17891	1.27964	1.38869	1.50673	1.63448	1.77270
34	1.18480	1.28923	1.40258	1.52557	1.65900	1.80373
35	1.19073	1.29890	1.41660	1.54464	1.68388	1.83529
36	1.19668	1.30865	1.43077	1.56394	1.70914	1.86741
37	1.20266	1.31846	1.44508	1.58349	1.73478	1.90009
38	1.20868	1.32835	1.45953	1.60329	1.76080	1.93334
39	1.21472	1.33831	1.47412	1.62333	1.78721	1.96717
40	1.22079	1.34835	1.48886	1.64362	1.81402	2.00160

TABLE LXXXIX (continued)

FUTURE VALUE OF $1

$s = (1+i)^n$

2%	$2\frac{1}{2}$%	3%	$3\frac{1}{2}$%	4%	5%	n
1.02000	1.02500	1.03000	1.03500	1.04000	1.05000	1
1.04040	1.05063	1.06090	1.07123	1.08160	1.10250	2
1.06121	1.07689	1.09273	1.10872	1.12486	1.15763	3
1.08243	1.10381	1.12551	1.14752	1.16986	1.21551	4
1.10408	1.13141	1.15927	1.18769	1.21665	1.27628	5
1.12616	1.15969	1.19405	1.22926	1.26532	1.34010	6
1.14869	1.18869	1.22987	1.27228	1.31593	1.40710	7
1.17166	1.21840	1.26677	1.31681	1.36857	1.47746	8
1.19509	1.24886	1.30477	1.36290	1.42331	1.55133	9
1.21899	1.28009	1.34392	1.41060	1.48024	1.62890	10
1.24337	1.31209	1.38423	1.45997	1.53945	1.71034	11
1.26824	1.34489	1.42576	1.51107	1.60103	1.79586	12
1.29361	1.37851	1.46853	1.56396	1.66507	1.88565	13
1.31948	1.41297	1.51259	1.61870	1.73168	1.97993	14
1.34587	1.44830	1.55797	1.67535	1.80094	2.07893	15
1.37279	1.48451	1.60471	1.73400	1.87298	2.18288	16
1.40024	1.52162	1.65285	1.79468	1.94790	2.29202	17
1.42825	1.55966	1.70243	1.85749	2.02582	2.40662	18
1.45681	1.59865	1.75351	1.92250	2.10685	2.52695	19
1.48595	1.63862	1.80611	1.98979	2.19112	2.65330	20
1.51567	1.67958	1.86030	2.05943	2.27877	2.78596	21
1.54598	1.72157	1.91610	2.13151	2.36992	2.92526	22
1.57690	1.76461	1.97359	2.20612	2.46472	3.07152	23
1.60844	1.80873	2.03279	2.28333	2.56330	3.22510	24
1.64061	1.85394	2.09378	2.36325	2.66584	3.38636	25
1.67342	1.90029	2.15659	2.44596	2.77247	3.55567	26
1.70689	1.94780	2.22129	2.53157	2.88337	3.73346	27
1.74102	1.99650	2.28793	2.62017	2.99870	3.92013	28
1.77585	2.04641	2.35657	2.71188	3.11865	4.11614	29
1.81136	2.09757	2.42726	2.80679	3.24340	4.32194	30
1.84759	2.15001	2.50008	2.90503	3.37313	4.53804	31
1.88454	2.20376	2.57508	3.00671	3.50806	4.76494	32
1.92223	2.25885	2.65234	3.11194	3.64838	5.00319	33
1.96068	2.31532	2.73191	3.22086	3.79432	5.25335	34
1.99989	2.37321	2.81386	3.33359	3.94609	5.51602	35
2.03989	2.43254	2.89828	3.45027	4.10393	5.79182	36
2.08069	2.49335	2.98523	3.57103	4.26809	6.08141	37
2.12230	2.55568	3.07478	3.69601	4.43881	6.38548	38
2.16475	2.61958	3.16703	3.82537	4.61637	6.70475	39
2.20804	2.68506	3.26204	3.95926	4.80102	7.03999	40

TABLE XC

PRESENT VALUE OF $1

$$v_{\overline{i}}^{n} = (1+i)^{-n} = \frac{1}{(1+i)^n}$$

n	$\frac{1}{2}\%$	$\frac{3}{4}\%$	1%	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
1	0.99503	0.99256	0.99010	0.98764	0.98522	0.98280
2	.99008	.98517	.98030	.97546	.97066	.96590
3	.98515	.97783	.97059	.96342	.95632	.94929
4	.98025	.97055	.96098	.95152	.94218	.93296
5	.97537	.96333	.95147	.93978	.92826	.91691
6	.97052	.95616	.94205	.92818	.91454	.90114
7	.96569	.94904	.93272	.91672	.90103	.88564
8	.96089	.94198	.92348	.90540	.88771	.87041
9	.95611	.93496	.91434	.89422	.87459	.85544
10	.95135	.92800	.90529	.88318	.86167	.84073
11	.94662	.92110	.89632	.87228	.84893	.82627
12	.94191	.91424	.88745	.86151	.83639	.81206
13	.93722	.90743	.87866	.85087	.82403	.79809
14	.93256	.90068	.86996	.84037	.81185	.78437
15	.92792	.89397	.86135	.82999	.79985	.77088
16	.92330	.88732	.85282	.81975	.78803	.75762
17	.91871	.88071	.84438	.80963	.77639	.74459
18	.91414	.87416	.83602	.79963	.76491	.73178
19	.90959	.86765	.82774	.78976	.75361	.71919
20	.90506	.86119	.81955	.78001	.74247	.70683
21	.90056	.85478	.81143	.77038	.73150	.69467
22	.89608	.84842	.80340	.76087	.72069	.68272
23	.89162	.84210	.79544	.75148	.71004	.67098
24	.88719	.83583	.78757	.74220	.69954	.65944
25	.88277	.82961	.77977	.73303	.68921	.64810
26	.87838	.82343	.77205	.72398	.67902	.63695
27	.87401	.81730	.76440	.71505	.66899	.62600
28	.86966	.81122	.75684	.70622	.65910	.61523
29	.86534	.80518	.74934	.69750	.64936	.60465
30	.86103	.79919	.74192	.68889	.63976	.59425
31	.85675	.79324	.73458	.68038	.63031	.58403
32	.85248	.78733	.72730	.67198	.62099	.57398
33	.84824	.78147	.72010	.66369	.61182	.56411
34	.84402	.77565	.71297	.65549	.60277	.55441
35	.83982	.76988	.70591	.64740	.59387	.54487
36	.83565	.76415	.69893	.63941	.58509	.53550
37	.83149	.75846	.69201	.63152	.57644	.52629
38	.82735	.75281	.68515	.62372	.56792	.51724
39	.82324	.74721	.67837	.61602	.55953	.50834
40	.81914	.74165	.67165	.60841	.55126	.49960

TABLE XC (continued)

PRESENT VALUE OF $1

$$v_{\bar{i}}^{n} = (1+i)^{-n} = \frac{1}{(1+i)^{n}}$$

2%	$2\frac{1}{2}$%	3%	$3\frac{1}{2}$%	4%	5%	n
0.98039	0.97561	0.97087	0.96618	0.96154	0.95238	1
.96117	.95181	.94260	.93351	.92456	.90703	2
.94232	.92860	.91514	.90194	.88900	.86384	3
.92385	.90595	.88849	.87144	.85480	.82270	4
.90573	.88385	.86261	.84197	.82193	.78353	5
.88797	.86230	.83748	.81350	.79032	.74622	6
.87056	.84127	.81309	.78599	.75992	.71068	7
.85349	.82075	.78941	.75941	.73069	.67684	8
.83676	.80073	.76642	.73373	.70259	.64461	9
.82035	.78120	.74409	.70892	.67556	.61391	10
.80426	.76215	.72242	.68495	.64958	.58468	11
.78849	.74356	.70138	.66178	.62460	.55684	12
.77303	.72542	.68095	.63940	.60057	.53032	13
.75788	.70773	.66112	.61778	.57748	.50507	14
.74302	.69047	.64186	.59689	.55527	.48102	15
.72845	.67363	.62317	.57671	.53391	.45811	16
.71416	.65720	.60502	.55720	.51337	.43630	17
.70016	.64117	.58740	.53836	.49363	.41552	18
.68643	.62553	.57029	.52016	.47464	.39573	19
.67297	.61027	.55368	.50257	.45639	.37689	20
.65978	.59539	.53755	.48557	.43883	.35894	21
.64684	.58087	.52189	.46915	.42196	.34185	22
.63416	.56670	.50669	.45329	.40573	.32557	23
.62172	.55288	.49193	.43796	.39012	.31007	24
.60953	.53939	.47761	.42315	.37512	.29530	25
.59758	.52624	.46370	.40884	.36069	.28124	26
.58586	.51340	.45019	.39501	.34682	.26785	27
.57438	.50088	.43708	.38165	.33348	.25509	28
.56311	.48866	.42435	.36875	.32065	.24295	29
.55207	.47674	.41199	.35628	.30832	.23138	30
.54125	.46512	.39999	.34423	.29646	.22036	31
.53063	.45377	.38834	.33259	.28506	.20987	32
.52023	.44270	.37703	.32134	.27409	.19987	33
.51003	.43191	.36605	.31048	.26355	.19036	34
.50003	.42137	.35538	.29998	.25342	.18129	35
.49022	.41109	.34503	.28983	.24367	.17266	36
.48061	.40107	.33498	.28003	.23430	.16444	37
.47119	.39129	.32523	.27056	.22529	.15661	38
.46195	.38174	.31575	.26141	.21662	.14915	39
.45289	.37243	.30656	.25257	.20829	.14205	40

TABLE XCI

FUTURE VALUE OF A PAYMENT OF $1 PER PERIOD

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} = \frac{s-1}{i}$$

n	$\frac{1}{2}\%$	$\frac{3}{4}\%$	1%	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	2.00500	2.00750	2.01000	2.01250	2.01500	2.01750
3	3.01503	3.02256	3.03010	3.03766	3.04523	3.05281
4	4.03010	4.04523	4.06040	4.07563	4.09090	4.10623
5	5.05025	5.07557	5.10101	5.12657	5.15227	5.17809
6	6.07550	6.11363	6.15202	6.19065	6.22955	6.26871
7	7.10588	7.15948	7.21354	7.26804	7.32299	7.37841
8	8.14141	8.21318	8.28567	8.35889	8.43284	8.50753
9	9.18212	9.27478	9.36853	9.46337	9.55933	9.65641
10	10.22803	10.34434	10.46221	10.58167	10.70272	10.82540
11	11.27917	11.42192	11.56683	11.71394	11.86326	12.01484
12	12.33556	12.50759	12.68250	12.86036	13.04121	13.22510
13	13.39724	13.60139	13.80933	14.02112	14.23683	14.45654
14	14.46423	14.70340	14.94742	15.19638	15.45038	15.70953
15	15.53655	15.81368	16.09690	16.38634	16.68214	16.98445
16	16.61423	16.93228	17.25787	17.59116	17.93237	18.28168
17	17.69730	18.05927	18.43044	18.81105	19.20136	19.60161
18	18.78579	19.19472	19.61475	20.04619	20.48938	20.94464
19	19.87972	20.33868	20.81090	21.29677	21.79672	22.31117
20	20.97912	21.49122	22.01900	22.56298	23.12367	23.70161
21	22.08401	22.65240	23.23919	23.84502	24.47052	25.11639
22	23.19443	23.82230	24.47159	25.14308	25.83758	26.55593
23	24.31040	25.00096	25.71630	26.45737	27.22514	28.02066
24	25.43196	26.18847	26.97347	27.78808	28.63352	29.51102
25	26.55912	27.38488	28.24320	29.13544	30.06302	31.02746
26	27.69191	28.59027	29.52563	30.49963	31.51397	32.57044
27	28.83037	29.80470	30.82089	31.88087	32.98668	34.14042
28	29.97452	31.02823	32.12910	33.27938	34.48148	35.73788
29	31.12440	32.26095	33.45039	34.69538	35.99870	37.36329
30	32.28002	33.50290	34.78489	36.12907	37.53868	39.01715
31	33.44142	34.75417	36.13274	37.58068	39.10176	40.69995
32	34.60862	36.01483	37.49407	39.05044	40.68829	42.41220
33	35.78167	37.28494	38.86901	40.53857	42.29861	44.15441
34	36.96058	38.56458	40.25770	42.04530	43.93309	45.92712
35	38.14538	39.85381	41.66028	43.57087	45.59209	47.73084
36	39.33611	41.15272	43.07688	45.11551	47.27597	49.56613
37	40.53279	42.46136	44.50765	46.67945	48.98511	51.43354
38	41.73545	43.77982	45.95272	48.26294	50.71989	53.33362
39	42.94413	45.10817	47.41225	49.86623	52.48068	55.26696
40	44.15885	46.44648	48.88637	51.48956	54.26789	57.23413

TABLE XCI (continued)

FUTURE VALUE OF A PAYMENT OF $1 PER PERIOD

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} = \frac{s-1}{i}$$

2%	$2\frac{1}{2}$%	3%	$3\frac{1}{2}$%	4%	5%	n
1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
2.02000	2.02500	2.03000	2.03500	2.04000	2.05000	2
3.06040	3.07563	3.09090	3.10623	3.12160	3.15250	3
4.12161	4.15252	4.18363	4.21494	4.24646	4.31013	4
5.20404	5.25633	5.30914	5.36247	5.41632	5.52563	5
6.30812	6.38774	6.46841	6.55015	6.63298	6.80191	6
7.43428	7.54743	7.66246	7.77941	7.89829	8.14201	7
8.58297	8.73612	8.89234	9.05169	9.21423	9.54911	8
9.75463	9.95452	10.15911	10.36850	10.58280	11.02656	9
10.94972	11.20338	11.46388	11.73139	12.00611	12.57789	10
12.16872	12.48347	12.80780	13.14199	13.48635	14.20679	11
13.41209	13.79555	14.19203	14.60196	15.02581	15.91713	12
14.68033	15.14044	15.61779	16.11303	16.62684	17.71298	13
15.97394	16.51895	17.08632	17.67699	18.29191	19.59863	14
17.29342	17.93193	18.59891	19.29568	20.02359	21.57856	15
18.63929	19.38023	20.15688	20.97103	21.82453	23.65749	16
20.01207	20.86473	21.76159	22.70502	23.69751	25.84037	17
21.41231	22.38635	23.41444	24.49969	25.64541	28.13239	18
22.84056	23.94601	25.11687	26.35718	27.67123	30.53900	19
24.29737	25.54466	26.87037	28.27968	29.77808	33.06595	20
25.78332	27.18327	28.67649	30.26947	31.96920	35.71925	21
27.29898	28.86286	30.53678	32.32890	34.24797	38.50521	22
28.84496	30.58443	32.45288	34.46041	36.61789	41.43048	23
30.42186	32.34904	34.42647	36.66653	39.08260	44.50200	24
32.03030	34.15776	36.45926	38.94986	41.64591	47.72710	25
33.67091	36.01171	38.55304	41.31310	44.31175	51.11345	26
35.34432	37.91200	40.70963	43.75906	47.08421	54.66913	27
37.05121	39.85980	42.93092	46.29063	49.96758	58.40258	28
38.79224	41.85630	45.21885	48.91080	52.96629	62.32271	29
40.56808	43.90270	47.57542	51.62268	56.08494	66.43885	30
42.37944	46.00027	50.00268	54.42947	59.32834	70.76078	31
44.22703	48.15028	52.50276	59.33450	62.70147	75.29883	32
46.11157	50.35404	55.07784	60.34121	66.20953	80.06377	33
48.03380	52.61289	57.73018	63.45315	69.85791	85.06696	34
49.99448	54.92821	60.46208	66.67401	73.65223	90.32031	35
51.99437	57.30141	63.27594	70.00760	77.59831	95.83632	36
54.03426	59.73395	66.17422	73.45787	81.70225	101.62814	37
56.11494	62.22730	69.15945	77.02890	85.97034	107.70955	38
58.23724	64.78298	72.23423	80.72491	90.40915	114.09502	39
60.40198	67.40255	75.40126	84.55028	95.02552	120.79977	40

TABLE XCII

PRESENT VALUE OF A PAYMENT OF $1 PER PERIOD

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i} = \frac{1-v^n_i}{i}$$

n	$\frac{1}{2}\%$	$\frac{3}{4}\%$	1%	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
1	0.99503	0.99256	0.99010	0.98765	0.98522	0.98280
2	1.98510	1.97772	1.97040	1.96312	1.95588	1.94870
3	2.97025	2.95556	2.94099	2.92653	2.91220	2.89798
4	3.95050	3.92611	3.90197	3.87806	3.85439	3.83094
5	4.92587	4.88944	4.85343	4.81784	4.78265	4.74786
6	5.89638	5.84560	5.79548	5.74601	5.69719	5.64900
7	6.86207	6.79464	6.72820	6.66273	6.59821	6.53464
8	7.82296	7.73661	7.65168	7.56812	7.48593	7.40505
9	8.77906	8.67158	8.56602	8.46235	8.36052	8.26049
10	9.73041	9.59958	9.47131	9.34553	9.22219	9.10122
11	10.67703	10.52068	10.36763	10.21780	10.07112	9.92749
12	11.61893	11.43491	11.25508	11.07931	10.90751	10.73955
13	12.55615	12.34235	12.13374	11.93019	11.73153	11.53764
14	13.48871	13.24302	13.00370	12.77055	12.54338	12.32201
15	14.41663	14.13700	13.86505	13.60055	13.34323	13.09288
16	15.33993	15.02431	14.71787	14.42029	14.13126	13.85050
17	16.25863	15.90503	15.56225	15.22992	14.90765	14.59508
18	17.17277	16.77918	16.39827	16.02955	15.67256	15.32686
19	18.08236	17.64683	17.22601	16.81931	16.42617	16.04606
20	18.98742	18.50802	18.04555	17.59932	17.16864	16.75288
21	19.88798	19.36280	18.85698	18.36970	17.90014	17.44755
22	20.78406	20.21122	19.66038	19.13056	18.62082	18.13027
23	21.67568	21.05332	20.45582	19.88204	19.33086	18.80125
24	22.56287	21.88915	21.24339	20.62424	20.03041	19.46069
25	23.44564	22.71876	22.02316	21.35727	20.71961	20.10878
26	24.32402	23.54219	22.79520	22.08125	21.39863	20.74573
27	25.19803	24.35949	23.55961	22.79630	22.06762	21.37173
28	26.06769	25.17071	24.31644	23.50252	22.72672	21.98696
29	26.93302	25.97589	25.06579	24.20002	23.37608	22.59160
30	27.79405	26.77508	25.80771	24.88891	24.01584	23.18585
31	28.65080	27.56832	26.54229	25.56929	24.64615	23.76988
32	29.50328	28.35565	27.26959	26.24127	25.26714	24.34386
33	30.35153	29.13712	27.98969	26.90496	25.87895	24.90797
34	31.19555	29.91278	28.70267	27.56046	26.48173	25.46238
35	32.03537	30.68266	29.40858	28.20786	27.07560	26.00725
36	32.87102	31.44681	30.10751	28.84727	27.66068	26.54275
37	33.70250	32.20527	30.79951	29.47878	28.23713	27.06905
38	34.52986	32.95808	31.48466	30.10250	28.80505	27.58629
39	35.35309	33.70529	32.16303	30.71852	29.36458	28.09463
40	36.17223	34.44694	32.83469	31.32693	29.91585	28.59423

TABLE XCII (continued)

PRESENT VALUE OF A PAYMENT OF $1 PER PERIOD

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i} = \frac{1-v^n_i}{i}$$

2%	$2\frac{1}{2}$%	3%	$3\frac{1}{2}$%	4%	5%	n
0.98039	0.97561	0.97087	0.96618	0.96154	0.95238	1
1.94156	1.92742	1.91347	1.89969	1.88610	1.85941	2
2.88388	2.85602	2.82861	2.80164	2.77509	2.72325	3
3.80773	3.76197	3.71710	3.67308	3.62990	3.54595	4
4.71346	4.64583	4.57971	4.51505	4.45182	4.32948	5
5.60143	5.50813	5.41719	5.32855	5.24214	5.07569	6
6.47199	6.34939	6.23028	6.11454	6.00206	5.78637	7
7.32548	7.17014	7.01969	6.87396	6.73275	6.46321	8
8.16224	7.97087	7.78611	7.60769	7.43533	7.10782	9
8.98259	8.75206	8.53020	8.31661	8.11090	7.72174	10
9.78685	9.51421	9.25262	9.00155	8.76048	8.30641	11
10.57534	10.25777	9.95400	9.66333	9.38507	8.86325	12
11.34837	10.98319	10.63496	10.30274	9.98565	9.39357	13
12.10625	11.69091	11.29607	10.92052	10.56312	9.89864	14
12.84926	12.38138	11.93794	11.51741	11.11839	10.37966	15
13.57771	13.05500	12.56110	12.09412	11.65230	10.83777	16
14.29187	13.71220	13.16612	12.65132	12.16567	11.27407	17
14.99203	14.35336	13.75351	13.18968	12.65930	11.68959	18
15.67846	14.97889	14.32380	13.70984	13.13394	12.08532	19
16.35143	15.58916	14.87748	14.21240	13.59033	12.46221	20
17.01121	16.18455	15.41502	14.69797	14.02916	12.82115	21
17.65805	16.76541	15.93692	15.16713	14.45112	13.16300	22
18.29220	17.33211	16.44361	15.62041	14.85684	13.48857	23
18.91393	17.88499	16.93554	16.05837	15.24696	13.79864	24
19.52346	18.42438	17.41315	16.48152	15.62208	14.09395	25
20.12104	18.95061	17.87684	16.89035	15.98277	14.37519	26
20.70690	19.46401	18.32703	17.28537	16.32959	14.64303	27
21.28127	19.96489	18.76411	17.66702	16.66306	14.89813	28
21.84439	20.45355	19.18846	18.03577	16.98372	15.14107	29
22.39646	20.93029	19.60044	18.39205	17.29203	15.37245	30
22.93770	21.39541	20.00043	18.73628	17.58849	15.59281	31
23.46834	21.84918	20.38877	19.06887	17.87355	15.80268	32
23.98856	22.29188	20.76579	19.39021	18.14765	16.00255	33
24.49859	22.72379	21.13184	19.70068	18.41120	16.19290	34
24.99862	23.14516	21.48722	20.00066	18.66461	16.37419	35
25.48884	23.55625	21.83225	20.29049	18.90828	16.54685	36
25.96945	23.95732	22.16724	20.57053	19.14258	16.71129	37
26.44064	24.34860	22.49246	20.84109	19.36786	16.86789	38
26.90259	24.73034	22.80822	21.10250	19.58449	17.01704	39
27.35548	25.10278	23.11477	21.35507	19.79277	17.15909	40

TABLE XCIII

SINKING FUNDS AND AMORTIZATION FACTORS

$$\text{Amortization Factor} = \frac{1}{a_{\overline{n}|i}}; \quad \text{Sinking-Fund Factor} = \frac{1}{s_{\overline{n}|i}} = \frac{1}{a_{\overline{n}|i}} - i$$

n	$\frac{1}{2}\%$	$\frac{3}{4}\%$	1%	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
1	1.00500	1.00750	1.01000	1.01250	1.01500	1.01750
2	0.50375	0.50563	0.50751	0.50939	0.51128	0.51316
3	.33667	.33835	.34002	.34170	.34338	.34507
4	.25313	.25471	.25628	.25786	.25945	.26103
5	.20301	.20452	.20604	.20756	.20909	.21062
6	.16960	.17107	.17255	.17403	.17553	.17702
7	.14573	.14718	.14863	.15009	.15156	.15303
8	.12783	.12926	.13069	.13213	.13358	.13504
9	.11391	.11532	.11674	.11817	.11961	.12106
10	.10277	.10417	.10558	.10700	.10843	.10988
11	.09366	.09505	.09645	.09787	.09929	.10073
12	.08607	.08745	.08885	.09026	.09168	.09311
13	.07964	.08102	.08242	.08382	.08524	.08667
14	.07414	.07551	.07690	.07831	.07972	.08116
15	.06936	.07074	.07212	.07353	.07494	.07638
16	.06519	.06656	.06795	.06935	.07077	.07220
17	.06151	.06287	.06426	.06566	.06708	.06852
18	.05823	.05960	.06098	.06239	.06381	.06525
19	.05530	.05667	.05805	.05946	.06088	.06232
20	.05267	.05403	.05542	.05682	.05825	.05969
21	.05028	.05165	.05303	.05444	.05587	.05732
22	.04811	.04948	.05086	.05227	.05370	.05516
23	.04614	.04750	.04889	.05030	.05173	.05319
24	.04432	.04569	.04707	.04849	.04992	.05139
25	.04265	.04402	.04541	.04682	.04826	.04973
26	.04111	.04248	.04387	.04529	.04673	.04820
27	.03969	.04105	.04245	.04387	.04532	.04679
28	.03836	.03973	.04112	.04255	.04400	.04548
29	.03713	.03850	.03990	.04132	.04278	.04426
30	.03598	.03735	.03875	.04018	.04164	.04313
31	.03490	.03627	.03768	.03911	.04057	.04207
32	.03390	.03527	.03667	.03811	.03958	.04108
33	.03295	.03432	.03573	.03717	.03864	.04015
34	.03206	.03343	.03484	.03628	.03776	.03927
35	.03122	.03259	.03400	.03545	.03693	.03845
36	.03042	.03180	.03321	.03467	.03615	.03768
37	.02967	.03105	.03247	.03392	.03541	.03694
38	.02896	.03034	.03176	.03322	.03472	.03625
39	.02829	.02967	.03109	.03255	.03406	.03559
40	.02765	.02903	.03046	.03192	.03343	.03497

TABLE XCIII (continued)

SINKING FUNDS AND AMORTIZATION FACTORS

$$\text{Amortization Factor} = \frac{1}{a_{\overline{n}|i}}; \quad \text{Sinking-Fund Factor} = \frac{1}{s_{\overline{n}|i}} = \frac{1}{a_{\overline{n}|i}} - i$$

2%	$2\frac{1}{2}$%	3%	$3\frac{1}{2}$%	4%	5%	n
1.02000	1.02500	1.03000	1.03500	1.04000	1.05000	1
0.51505	0.51883	0.52261	0.52640	0.53020	0.53781	2
.34676	.35014	.35353	.35693	.36035	.36721	3
.26262	.26582	.26903	.27225	.27549	.28201	4
.21216	.21525	.21836	.22148	.22463	.23098	5
.17853	.18155	.18460	.18767	.19076	.19702	6
.15451	.15750	.16051	.16355	.16661	.17282	7
.13651	.13947	.14246	.14548	.14853	.15472	8
.12252	.12546	.12843	.13145	.13449	.14069	9
.11133	.11426	.11723	.12024	.12329	.12951	10
.10218	.10511	.10808	.11109	.11415	.12039	11
.09456	.09749	.10046	.10348	.10655	.11283	12
.08812	.09105	.09403	.09706	.10014	.10646	13
.08260	.08554	.08853	.09157	.09467	.10102	14
.07783	.08077	.08377	.08683	.08994	.09634	15
.07365	.07660	.07961	.08269	.08582	.09227	16
.06997	.07293	.07595	.07904	.08220	.08870	17
.06670	.06967	.07271	.07582	.07899	.08555	18
.06378	.06676	.06981	.07294	.07614	.08275	19
.06116	.06415	.06722	.07036	.07358	.08024	20
.05879	.06179	.06487	.06804	.07128	.07800	21
.05663	.05965	.06275	.06593	.06920	.07597	22
.05467	.05770	.06081	.06402	.06731	.07414	23
.05287	.05591	.05905	.06227	.06559	.07247	24
.05122	.05428	.05743	.06067	.06401	.07095	25
.04970	.05277	.05594	.05921	.06257	.06956	26
.04829	.05138	.05456	.05785	.06124	.06829	27
.04699	.05009	.05329	.05660	.06001	.06712	28
.04578	.04889	.05212	.05545	.05888	.06605	29
.04465	.04778	.05102	.05437	.05783	.06505	30
.04360	.04674	.05000	.05337	.05686	.06413	31
.04261	.04577	.04905	.05244	.05595	.06328	32
.04169	.04486	.04816	.05157	.05510	.06249	33
.04082	.04401	.04732	.05076	.05432	.06176	34
.04000	.04321	.04654	.05000	.05358	.06107	35
.03923	.04245	.04580	.04928	.05289	.06044	36
.03851	.04174	.04511	.04861	.05224	.05984	37
.03782	.04107	.04446	.04798	.05163	.05928	38
.03717	.04044	.04384	.04739	.05106	.05877	39
.03656	.03984	.04326	.04683	.05052	.05828	40

TABLE XCIV

FUTURE VALUE OF $1 FOR FRACTIONAL PART OF A PERIOD

$$s_{\overline{p}|} = (1+i)^{\frac{1}{p}}$$

p	$\frac{1}{2}\%$	$\frac{3}{4}\%$	1%	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
2	1.00250	1.00374	1.00499	1.00623	1.00747	1.00871
3	1.00166	1.00249	1.00332	1.00415	1.00498	1.00580
4	1.00125	1.00187	1.00249	1.00311	1.00373	1.00435
6	1.00083	1.00125	1.00166	1.00207	1.00249	1.00290
12	1.00042	1.00062	1.00083	1.00104	1.00124	1.00145

p	2%	$2\frac{1}{2}\%$	3%	$3\frac{1}{2}\%$	4%	5%
2	1.00995	1.01242	1.01489	1.01735	1.01980	1.02470
3	1.00662	1.00827	1.00990	1.01153	1.01316	1.01640
4	1.00496	1.00619	1.00742	1.00864	1.00985	1.01227
6	1.00331	1.00412	1.00494	1.00575	1.00656	1.00817
12	1.00165	1.00206	1.00247	1.00287	1.00327	1.00407

TABLE XCV

DECIMAL EQUIVALENTS OF A POUND IN SHILLINGS AND PENCE

£	s.d.	£	s.d.	£	s.d.	£	s.d.	£	s.d.
0.00417	0/1	0.10417	2/1	0.20417	4/1	0.30417	6/1	0.40417	8/1
.00833	0/2	.10833	2/2	.20833	4/2	.30833	6/2	.40833	8/2
.01250	0/3	.11250	2/3	.21250	4/3	.31250	6/3	.41250	8/3
.01667	0/4	.11667	2/4	.21667	4/4	.31667	6/4	.41667	8/4
.02083	0/5	.12083	2/5	.22083	4/5	.32083	6/5	.42083	8/5
.02500	0/6	.12500	2/6	.22500	4/6	.32500	6/6	.42500	8/6
.02917	0/7	.12917	2/7	.22917	4/7	.32917	6/7	.42917	8/7
.03333	0/8	.13333	2/8	.23333	4/8	.33333	6/8	.43333	8/8
.03750	0/9	.13750	2/9	.23750	4/9	.33750	6/9	.43750	8/9
.04167	0/10	.14167	2/10	.24167	4/10	.34167	6/10	.44167	8/10
.04583	0/11	.14583	2/11	.24583	4/11	.34583	6/11	.44583	8/11
.05000	1/0	.15000	3/0	.25000	5/0	.35000	7/0	.45000	9/0
.05417	1/1	.15417	3/1	.25417	5/1	.35417	7/1	.45417	9/1
.05833	1/2	.15833	3/2	.25833	5/2	.35833	7/2	.45833	9/2
.06250	1/3	.16250	3/3	.26250	5/3	.36250	7/3	.46250	9/3
.06667	1/4	.16667	3/4	.26667	5/4	.36667	7/4	.46667	9/4
.07083	1/5	.17083	3/5	.27083	5/5	.37083	7/5	.47083	9/5
.07500	1/6	.17500	3/6	.27500	5/6	.37500	7/6	.47500	9/6
.07917	1/7	.17917	3/7	.27917	5/7	.37917	7/7	.47917	9/7
.08333	1/8	.18333	3/8	.28333	5/8	.38333	7/8	.48333	9/8
.08750	1/9	.18750	3/9	.28750	5/9	.38750	7/9	.48750	9/9
.09167	1/10	.19167	3/10	.29167	5/10	.39167	7/10	.49167	9/10
.09583	1/11	.19583	3/11	.29583	5/11	.39583	7/11	.49583	9/11
.10000	2/0	.20000	4/0	.30000	6/0	.40000	8/0	.50000	10/0

For values beyond 10*s*., deduct 10*s*. from the amount and look up the decimal equivalent of the remainder. Add the decimal equivalent of the remainder to the decimal equivalent of 10*s*.

TABLE XCVI
SPRAGUE TABLES

Values, to the Nearest Cent, of a Bond for $1,000,000 at 5% Interest, Payable Semiannually

Net Income	13 Years	13½ Years	14 Years	14½ Years	15 Years	15½ Years
2 50	1 276 015 66	1 284 953 74	1 293 781 47	1 302 500 22	1 311 111 33	1 319 616 13
2 55	1 269 642 20	1 278 343 32	1 286 934 90	1 295 418 32	1 303 794 94	1 312 066 09
2 60	1 263 307 30	1 271 774 24	1 280 132 51	1 288 383 53	1 296 528 66	1 304 569 26
2 65	1 257 010 71	1 265 246 20	1 273 373 99	1 281 395 50	1 289 312 12	1 297 125 21
2 70	1 250 752 18	1 258 758 93	1 266 659 03	1 274 453 91	1 282 144 95	1 289 733 55
2 75	1 244 531 45	1 253 312 16	1 259 987 33	1 267 558 40	1 275 026 79	1 282 393 87
2 80	1 238 348 28	1 245 905 60	1 253 358 58	1 260 708 66	1 267 957 26	1 275 105 78
2 85	1 232 202 43	1 239 539 00	1 246 772 49	1 253 904 35	1 260 936 02	1 267 868 88
2 90	1 226 093 65	1 233 212 07	1 240 228 75	1 247 145 15	1 253 962 69	1 260 682 79
2 95	1 220 021 69	1 226 924 55	1 233 727 08	1 240 430 72	1 247 036 93	1 253 547 11
3 00	1 213 986 32	1 220 676 17	1 227 267 17	1 233 760 76	1 240 158 38	1 246 461 46
3 05	1 207 987 29	1 214 466 68	1 220 848 73	1 227 134 92	1 233 326 69	1 239 425 45
3 10	1 202 024 38	1 208 295 79	1 214 471 48	1 220 552 91	1 226 541 52	1 232 438 72
3 15	1 196 097 33	1 202 163 26	1 208 135 13	1 214 014 41	1 219 802 52	1 225 500 88
3 20	1 190 205 93	1 196 068 83	1 201 839 40	1 207 519 09	1 213 109 34	1 218 611 56
3 25	1 184 349 93	1 190 012 23	1 195 583 99	1 201 066 66	1 206 461 66	1 211 770 39
3 30	1 178 529 11	1 183 993 22	1 189 368 64	1 194 656 80	1 199 859 12	1 204 977 00
3 35	1 172 743 23	1 178 011 53	1 183 193 05	1 188 289 21	1 193 301 41	1 198 231 04
3 40	1 166 992 06	1 172 066 93	1 177 056 96	1 181 963 58	1 186 788 18	1 191 532 13
3 45	1 161 275 39	1 166 159 15	1 170 960 09	1 175 679 61	1 180 319 11	1 184 879 93
3 50	1 155 592 99	1 160 287 95	1 164 902 16	1 169 437 01	1 173 893 87	1 178 274 07
3 55	1 149 944 62	1 154 453 08	1 158 882 91	1 163 235 48	1 167 512 14	1 171 714 21
3 60	1 144 330 08	1 148 654 30	1 152 902 07	1 157 074 72	1 161 173 60	1 165 200 00
3 65	1 138 749 14	1 142 891 37	1 146 959 36	1 150 954 45	1 154 877 92	1 158 731 08
3 70	1 133 201 58	1 137 164 05	1 141 054 54	1 144 874 36	1 148 624 80	1 152 307 12
3 75	1 127 687 18	1 131 472 08	1 135 187 32	1 138 834 18	1 142 413 92	1 145 927 77
3 80	1 122 205 74	1 125 815 25	1 129 357 46	1 132 833 62	1 136 244 96	1 139 592 70
3 85	1 116 757 03	1 120 193 31	1 123 564 69	1 126 872 39	1 130 117 63	1 133 301 57
3 90	1 111 340 84	1 114 606 02	1 117 808 75	1 120 950 22	1 124 031 60	1 127 054 05
3 95	1 105 956 96	1 109 053 16	1 112 089 39	1 115 066 82	1 117 986 59	1 120 849 80
4 00	1 100 605 18	1 103 534 49	1 106 406 36	1 109 221 92	1 111 982 28	1 114 688 51
4 05	1 095 285 29	1 098 049 78	1 100 759 41	1 103 415 25	1 106 018 38	1 108 569 84
4 10	1 089 997 09	1 092 598 82	1 095 148 28	1 097 646 52	1 100 094 58	1 102 493 47
4 15	1 084 740 37	1 087 181 36	1 089 572 72	1 091 915 48	1 094 210 61	1 096 459 08
4 20	1 079 514 92	1 081 797 18	1 084 032 50	1 086 221 84	1 088 366 15	1 090 466 36
4 25	1 074 320 55	1 076 446 07	1 078 527 37	1 080 565 35	1 082 560 93	1 084 514 99
4 30	1 069 157 05	1 071 127 80	1 073 057 08	1 074 945 74	1 076 794 66	1 078 604 66
4 35	1 064 024 22	1 065 842 15	1 067 621 39	1 069 362 75	1 071 067 04	1 072 735 05
4 40	1 058 921 86	1 060 588 91	1 062 220 07	1 063 816 11	1 065 377 80	1 066 905 87
4 45	1 053 849 78	1 055 367 85	1 056 852 87	1 058 305 57	1 059 726 65	1 061 116 81
4 50	1 048 807 78	1 050 178 76	1 051 519 57	1 052 830 87	1 054 113 32	1 055 367 55
4 55	1 043 795 67	1 045 021 43	1 046 219 92	1 047 391 76	1 048 537 53	1 049 657 82
4 60	1 038 813 24	1 039 895 64	1 040 953 71	1 041 987 98	1 042 999 01	1 043 987 30
4 65	1 033 860 32	1 034 801 19	1 035 720 68	1 036 619 28	1 037 497 47	1 038 355 70
4 70	1 028 936 70	1 029 737 86	1 030 520 63	1 031 285 42	1 032 032 65	1 032 762 73
4 75	1 024 042 21	1 024 705 45	1 025 353 31	1 025 986 14	1 026 604 29	1 027 208 10
4 80	1 019 176 64	1 019 703 75	1 020 218 51	1 020 721 20	1 021 212 11	1 021 691 51
4 85	1 014 339 83	1 014 732 56	1 015 116 00	1 015 490 36	1 015 855 85	1 016 212 70
4 90	1 009 531 57	1 009 791 67	1 010 045 56	1 010 293 37	1 010 535 26	1 010 771 36
4 95	1 004 751 69	1 004 880 89	1 005 006 96	1 005 130 00	1 005 250 06	1 005 367 22
5 00	1 000 000 00	1 000 000 00	1 000 000 00	1 000 000 00	1 000 000 00	1 000 000 00

TABLE XCVII

EQUITABLE BOND TABLES

20 Years

Interest Payable Semiannually

% Per Annum	3%	3½%	4%	4½%	5%	6%	7%
4.50	**80.3549**	**86.9032**	**93.4516**	**100.0000**	**106.5484**	**119.6451**	**132.7419**
	.050216	.033466	.016733	.....	.016733	.050216	.083666
	.001673	.001115	.000557	.....	.000557	.001673	.002788
4.55	**79.7870**	**86.3073**	**92.8276**	**99.3480**	**105.8683**	**118.9090**	**131.9496**
	.051366	.034783	0.18216	.001666	.014916	.048033	.081183
	.001712	.001159	.000607	.000055	.000497	.001601	.002706
4.60	**79.2241**	**85.7166**	**92.2090**	**98.7015**	**105.1940**	**118.1789**	**131.1638**
	.052483	.036083	.019666	.003283	.013116	.045916	.078733
	.001749	.001202	.000655	.000109	.000437	.001530	.002624
4⅝	**78.9446**	**85.4232**	**91.9018**	**98.3804**	**104.8589**	**117.8161**	**130.7733**
	.053033	.036716	.020400	.004083	.012250	.044883	.077516
	.001767	.001223	.000680	.000136	.000408	.001496	.002583
4.65	**78.6663**	**85.1310**	**91.5958**	**98.0606**	**104.5253**	**117.4549**	**130.3844**
	.053600	.037350	.021116	.004883	.011366	.043833	.076316
	.001786	.001245	.000703	.000162	.000378	.001461	.002543
4.70	**78.1134**	**84.5506**	**90.9879**	**97.4251**	**103.8623**	**116.7368**	**129.6113**
	.054666	.038583	.022516	.006433	.009650	.041800	.073950
	.001822	.001286	.000750	.000214	.000321	.001393	.002465
4.75	**77.5654**	**83.9753**	**90.3852**	**96.7951**	**103.2049**	**116.0247**	**128.8445**
	.055716	.039800	.023883	.007966	.007966	.039800	.074633
	.001857	.001326	.000796	.000265	.000265	.001326	.002387
4.80	**77.0222**	**83.4049**	**89.7877**	**96.1704**	**102.5531**	**115.3185**	**128.0840**
	.056733	.049966	.025216	.000466	.006300	.037816	.069333
	.001891	.001365	.000840	.000315	.000210	.001260	.002311
4.85	**76.4839**	**82.8396**	**89.1953**	**95.5510**	**101.9067**	**114.6181**	**127.3295**
	.057733	.042116	.026516	.010916	.004683	.035883	.067083
	.001924	.001403	.000883	.000363	.000156	.001196	.002236
4⅞	**76.2165**	**82.5588**	**88.9010**	**95.2433**	**101.5856**	**114.2701**	**126.9546**
	.058216	.042700	.027166	.011650	.003883	.034933	.065983
	.001940	.001423	.000905	.000388	.000129	.001164	.002199
4.90	**75.9503**	**82.2792**	**88.6080**	**94.9369**	**101.2658**	**113.9235**	**126.5813**
	.058700	.043250	.027800	.012250	.003083	.033983	.064866
	.001956	.001441	.000926	.000411	.000102	.001132	.002162
4.95	**75.4214**	**81.7236**	**88.0258**	**94.3280**	**100.6302**	**113.2346**	**125.8390**
	.059633	.044350	.029950	.013766	.001533	.032116	.062700
	.001987	.001478	.000968	.000458	.000051	.001070	.002090
5.00	**74.8972**	**81.1729**	**87.4486**	**93.7243**	**100.0000**	**112.5514**	**125.1028**
	.060550	.045416	.030283	.015133	.....	.030283	.060550
	.002018	.001513	.001009	.000504	.....	.001009	.002018
5.05	**74.3776**	**80.6270**	**86.8764**	**93.1257**	**99.3751**	**111.8738**	**124.3725**
	.061450	.046466	.031483	.016483	.001500	.028466	.058450
	.002048	.001548	.001049	.000549	.000050	.000948	.001948
5.10	**73.8627**	**80.0858**	**86.3090**	**92.5322**	**98.7554**	**111.2017**	**123.6481**
	.062333	.047483	.032650	.017816	.002966	.026716	.056383
	.002077	.001582	.001088	.000593	.000098	.000890	.001879
5⅛	**73.6069**	**79.8170**	**86.0272**	**92.2373**	**98.4475**	**110.8678**	**123.2881**
	.062766	.047983	.033233	.018450	.003700	.025833	.055366
	.002092	.001599	.001107	.000615	.000123	.000861	.001845
5.15	**73.3522**	**79.5494**	**85.7465**	**91.9437**	**98.1409**	**110.5352**	**122.9295**
	.063183	.048500	.033783	.019100	.004416	.024966	.054366
	.002106	.001616	.001126	.000636	.000147	.000832	.001812
5.20	**72.8462**	**79.0175**	**85.1889**	**91.3602**	**97.5315**	**109.8741**	**122.2167**
	.064000	.049450	.034916	.020366	.005816	.023266	.052366
	.002133	.001648	.001163	.000678	.000193	.000775	.001745
5.25	**72.3447**	**78.4903**	**84.6359**	**90.7816**	**96.9272**	**109.2184**	**121.5097**
	.064800	.050400	.036000	.021600	.007200	.021600	.050400
	.002160	.001680	.001200	.000720	.000240	.000720	.001680

TABLE XCVIII

ACCRUED DIVIDEND TABLES

365 days = 1 year

2% Days	2½% Days	3% Days	3½% Days	4% Days	4½% Days	5% Days	6% Days	7% Days	Interest on $100
23	18	15	13	11	10	9	8	7	$\frac{1}{8}$
46	36	30	26	23	20	18	15	13	$\frac{1}{4}$
68	55	46	39	34	30	27	23	20	$\frac{3}{8}$
91	73	61	52	46	40	37	30	26	$\frac{1}{2}$
114	91	76	65	57	51	46	38	33	$\frac{5}{8}$
137	109	91	78	68	61	55	46	39	$\frac{3}{4}$
160	128	107	91	80	71	64	53	46	$\frac{7}{8}$
183	146	122	104	91	81	73	61	52	1
	164	137	117	103	91	82	68	59	$1\frac{1}{8}$
	182	152	130	114	101	91	76	65	$1\frac{1}{4}$
		167	143	125	112	100	84	72	$1\frac{3}{8}$
		183	156	137	122	110	91	78	$1\frac{1}{2}$
			169	148	132	119	99	85	$1\frac{5}{8}$
			182	160	142	128	106	91	$1\frac{3}{4}$
				171	152	137	114	98	$1\frac{7}{8}$
				183	162	146	122	104	2
					172	155	129	111	$2\frac{1}{8}$
					183	164	137	117	$2\frac{1}{4}$
						173	144	124	$2\frac{3}{8}$
						183	152	130	$2\frac{1}{2}$
							160	137	$2\frac{5}{8}$
							167	143	$2\frac{3}{4}$
							175	150	$2\frac{7}{8}$
							183	156	3
								163	$3\frac{1}{8}$
								169	$3\frac{1}{4}$

Solutions to Problems and Exercises

Answers achieved by readers may differ by a cent or two in some instances if different methods of calculation are employed. Such discrepancies, since they are minor, should not be a cause of concern.

INSTALLMENT FINANCE

1 $2,599.94 **2** $2,568.65 or $2,847.18
3 $34,687.00 **4** $5,707.14 **5** $31,446.81
6 $6,370.59 **7** $2,915.15 **8** $231.80
9 $165.34 **10** $127.68; $2,247.08
11 $1,371.84 **12** $191.28
13 35 regular payments and $731 at time of 35th payment. If funds accumulated after 35 payments are allowed to draw interest for one more period, more than $200,000 will be available in the fund.
14 31 quarters or $7\frac{3}{4}$ years, $5,016.21
15 60 regular payments and a final payment of $38.42
16 21 regular and 1 partial payment
17 21 regular and 1 partial payment
18 102 regular payments and 1 partial payment
19 2% per annum **20** $\frac{2}{3}$% per quarter
21 3% per annum **22** 3.0665% per month
23 $\frac{7}{12}\% > i > \frac{1}{2}\%$ per month
24 0.90937% per month or 10.91244% per annum converted monthly
25 $131.47 **26** 15.45% per annum effective
27 $8,491.51
28 9 regular plus 1 partial payment
29 $4,070.36
30 By periods: 1st, $164.36; 2d, $328.72; 3d, $493.08; 4th, $739.62; 5th, $924.53
31 $157.64 **32** $14,395.53 **33** $406.63
34 $183.32
35 Same. Two algebraic expressions for the future value of an annuity due of 20 payments of $1 per period at 4% per period.
36 $126.91 **37** 60 regular plus 1 partial payment
38 $8\frac{1}{2}\% > i > 8\%$ **39** $179.03
40 53 regular payments and 1 partial payment
41 $2,452.70
42 (a)$148,683.25 (b)$127,548.40
43 $70,177.00
44 (a)24 payments of $200; (b)17 payments of $300 and a partial payment of $113.
45 $79.90 **46** $40,377
47 (a) $33,333.33 (b) $40,000
(c) $50,000 (d) $100,000
48 $166,666.67
49 A $19.16, B $17.34. B should be selected.

MONEY AND BANKING

1 219.36 grains **2** $2.19 **3** 77.78 to 1
4 $8.2397; 1.026 grains **5** 92 **6** $671.24
7 £224 0*s*. 6*d*. **8** £1.64583 **9** £5.27917
10 £0.43333 **11** £1.04583 **12** £9.53750
13 £5.25

	£	*s*.	*d*.
14	5	4	10
15	211	10	0
16	9	19	11
17	23	12	6
18	1	0	2
19	0	6	0

20 2549.45 pesos
21

Pesos = Dollars		Dollars = Pesos	
1	0.237525	1	4.21008
2	0.475050	2	8.42017
3	0.712575	3	12.63025
4	0.950100	4	16.84033
5	1.187625	5	21.05042
6	1.425150	6	25.26050
7	1.662675	7	29.47058
8	1.900200	8	33.68066
9	2.137725	9	37.89075

22 $87.65
23 3,586.99 pesos **24** £2 2*s*. 1*d*.
25 £1 2*s*. 1*d*. **26** (a) 4.41955 (b) 4.41911
27 On September 19, you should have sold sterling for $4.85\frac{7}{16}$ for delivery in a month. One month later, you could have bought in the spot market at $3.87 the necessary exchange to deliver to fulfill your contract.
28 (a) Buy futures in December at $4.91\frac{3}{8}$ for delivery in 90 days. Sell this exchange in the spot market in March at $4.99. (b) Sell 30-day futures in March at $4.98\frac{3}{4}$. Buy in spot market at 4.94 in April to fulfill contracts made in March.

MONEY AND BANKING (continued)

(c) Buy 90-day futures in April at $4.93\frac{1}{4}$. Sell this exchange in the spot market in July at $\$5.02\frac{1}{4}$.

29 PARITIES IN DOLLARS, LIRE, AND POUNDS STERLING

£	92	92¼	92½	92¾	93
	$1	$1	$1	$1	$1
4.65	5.05434	5.04065	5.02703	5.01348	5.00000
4.65¼	5.05706	5.04336	5.02973	5.01617	5.00269
4.65½	5.05978	5.04607	5.03243	5.01887	5.00538
4.65¾	5.06250	5.04878	5.03513	5.02156	5.00806

30 Not counting commissions, you would save $23.37 by buying pounds in London and converting them to lire. To pay direct would cost $12,625. To pay via London, the cost would be $12,601.63.

31 $22.77 gross profit

32 3.25 per cent **33** 3.117 per cent

34 4.899 **35** 4.7536 **36** 4.08355

37 4.46159 **38** 75.6372 **39** 103.88802

40 3.23 per cent

41 (a) 8.12%, 8.08%; (b) 6.07%, 6.035%

42 See foot of page.

43 $\frac{1}{4}$ per cent

44 The table shows that at 6% the dividend would accrue to $1\frac{1}{4}$ per cent in 76 days. The rate on this stock is 12 per cent; therefore it would accrue to $2\frac{1}{2}$ per cent in 76 days or approximately $\frac{3}{8}$ of a point per share. After deducting this $\frac{3}{8}$ from the price of the stock, it is still above the conversion parity; hence, conversion would be profitable.

45 (a) 16.84 (b) $28\frac{1}{2}$% (c) 1.34 (d) 0.65

46 (a) 21.978 (b) 19.80

47 Schedule of conversion parities of United Aircraft 5% cumulative convertible preferred stock convertible into common stock at $40 a share until January 1, 1952.

PREFERRED PRICES	COMMON PRICES	PREFERRED PRICES	COMMON PRICES
50	20	103¼	41.3
55	22	103½	41.4
60	24	103¾	41.5
65	26	104	41.6
70	28	104¼	41.7
75	30	104½	41.8
80	32	104¾	41.9
85	34	105	42
90	36	105¼	42.1
91	36.4	105½	42.2
92	36.8	105¾	42.3
93	37.2	106	42.4
94	37.6	106¼	42½
95	38	106½	42.6
96	38.4	106¾	42.7
97	38.8	107	42.8
98	39.2	107¼	42.9
99	39.6	107½	43
100	40	107¾	43.1
101	40.4	108	43.2
101¼	40½	108¼	43.3
101½	40.6	108½	43.4
101¾	40.7	108¾	43.5
102	40.8	109	43.6
102¼	40.9	109¼	43.7
102½	41	109½	43.8
102¾	41.1	109¾	43.9
103	41.2	110	44

48 1 per cent **49** $1\frac{3}{8}$ per cent

50 7.64 years **51** (a) $2.40 (b) $9.60

52 (a) $1.29 (b) $8.71

[42

	PRIOR DEDUCTION			CUMULATIVE			OVER-ALL		
	TIMES EARNED	FACTOR OF SAFETY	MARGIN OF SAFETY	TIMES EARNED	FACTOR OF SAFETY	MARGIN OF SAFETY	TIMES EARNED	FACTOR OF SAFETY	MARGIN OF SAFETY
1st Mtg. Bond	2	100%	50%	2	100%	50%	$1\frac{1}{4}$	25%	20%
1st Rfg. Bond	5	400%	80%	$1\frac{2}{3}$	$66\frac{2}{3}$%	40%	$1\frac{1}{4}$	25%	20%
Int. on Deb.	2	100%	50%	$1\frac{1}{4}$	25%	20%	$1\frac{1}{4}$	25%	20%

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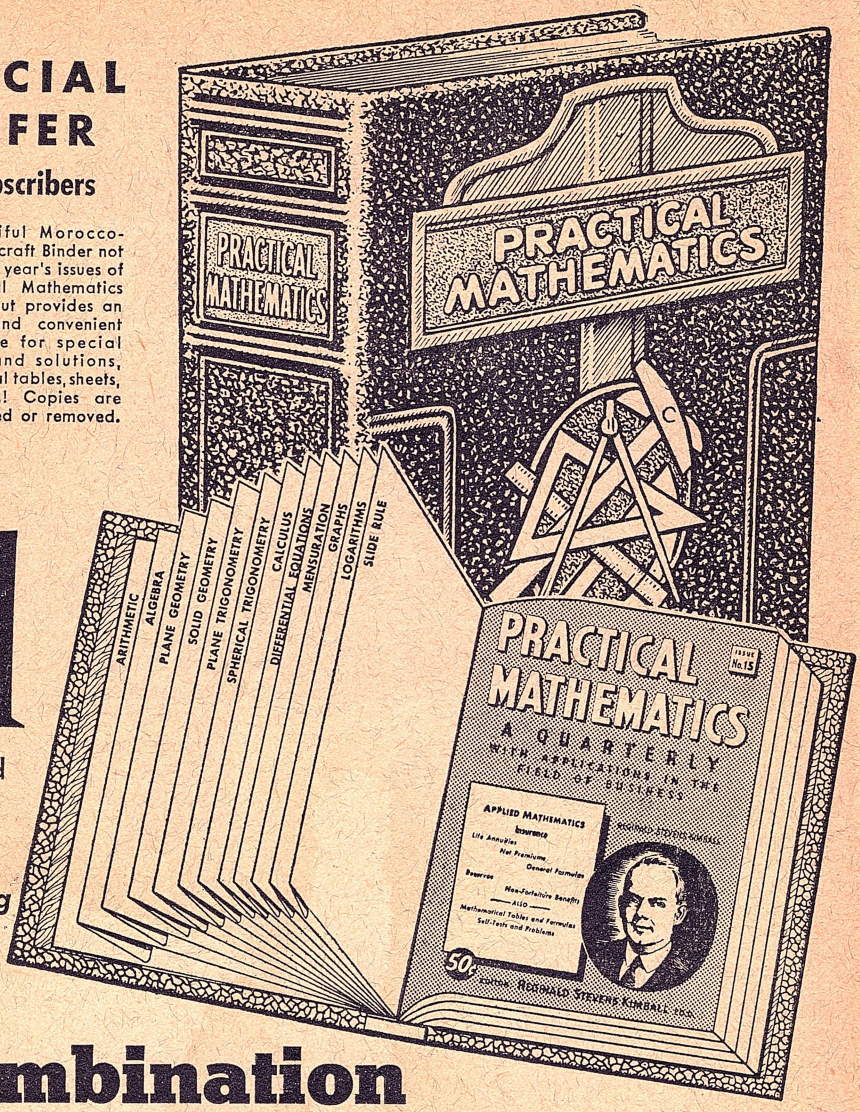
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